TIGHT BINDING BOOK

TOTAL TEXT FLY

ELEMENTARY

CALCULUS

BY

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PREFACE

This text on elementary differential and integral calculus is presented with the belief that it is adapted for use as an introductory course both in academic colleges and in engineering schools. The author has had over twenty years of experience in teaching the subject in colleges and universities using a great variety of standard texts. He has written this book as a result of that experience being convinced that the method of approach used is the most suitable for a first year calculus course.

Considerable attention has been given to the continuous development of the subject with special emphasis upon the necessity of understanding the fundamental principles. A conscious effort has been made to interest the student regardless of the reason for his election of the subject. It is recognized that his choice may be due to a desire to apply this most powerful tool of mathematics in some chosen field of science, or it may be due to an interest in the subject for its own sake. The language of the text is simple; full explanations and many illustrations are included.

This book provides unusual flexibility in the choice of material. The development is continuous through the first eleven chapters. Since the entire text cannot be taken in any one year, various selections of chapters are possible according to the individual requirements for a course. It is suggested that at least the first nine chapters form the nucleus of a course, supplementing this with selections from the remaining eight chapters. In this way provision has been made for those instructors who desire to give students having particular applications in mind, an introduction to such subjects as multiple integration, series, partial differentiation and differential equations.

Nearly all of the existing text-books divide the study of the calculus into separate parts, the differential and the integral calculus. In such books, all types of functions are differentiated or integrated, after which the applications are made.

Contrary to this method of study, the author has interwoven the processes of differentiation and integration. The first four chapters of this book form an introduction to the concepts, theories and applications of both differential and integral calculus in which the formal manipulation requires the use of polynomials in the majority of cases. The study of these

fundamental principles and applications is continued and extended by the introduction of new functions: algebraic, trigonometric, exponential and logarithmic. The author believes that a *continued* application of the calculus method to problems as they arise and require the use of more advanced functions, engenders a more complete understanding of the subject than the traditional method.

Another departure from the existing texts is that polar coordinates and their applications are presented in a separate chapter. The various sections may be used as parts of preceding chapters if desired. However, since many instructors prefer not to make use of rectangular and polar coordinates simultaneously, this chapter provides opportunity for a separate consideration of the latter.

Extreme care has been given to the preparation of Chapter VI. Classroom experience has led the author to feel that a complete understanding of the principles involved in this chapter is essential and that if these principles are mastered in applying the simpler functions, no difficulty should be encountered when the applications require the use of more advanced functions. The definite integral is defined by means of the indefinite integral. The fundamental theorem states that the limit of a sum is equal to the definite integral which is proved by means of the area under a curve. Detailed applications are given for various elements of area, volume, fluid pressure and work.

In an elementary calculus it is generally agreed that it is inadvisable to give rigorous proofs for all the theorems to which applications are made. In those cases in which proofs are omitted or are not complete, the assumptions are clearly stated. The student should be made to realize that there are proofs which are not within the scope of such a text and that such proofs exist in more advanced treatises.

A knowledge of plane analytic geometry is assumed and the content of that subject is used freely. A collection of formulas of trigonometry and plane analytic geometry is appended which may form a basis for a review of those subjects as needed.

Since a knowledge of solid analytic geometry is not generally presupposed, a chapter presenting the essentials of that subject precedes the calculus study of functions of two variables. Although this chapter is a condensation of the subject, it is felt that sufficient material is included for an understanding of the content of the two chapters which follow it.

An important feature of the book is the large number of graded exercises. These exercises give the instructor a wide range of choice as to number of problems and as to degree of difficulty. Also, by means of the exercises **a**

continuous review of previous chapters is provided. The answers are given for the odd-numbered problems only, giving the student opportunity to verify some of his results but placing him on his own resources in others.

I wish to express appreciation to my colleagues who have been good enough to give me the benefit of their teaching experience and who have given me many highly valuable criticisms. I am particularly indebted to Professors J. G. Hardy and D. E. Richmond for their reading of the manuscript and parts of the proof.

<div align="right">V. H. WELLS.</div>

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of x there corresponds one or more definite values of y, then y is said to be a function of x. If each of the above functions is represented by y, the functional relationships between x and y are given by the following equations:

$$y = 2x^2 + 1, \qquad y = \pm\sqrt{a^2 - x^2},$$

$$y = \frac{x}{x - a}, \qquad y = \sin x,$$

$$y = \arctan x, \qquad y = \log x.$$

Dependent and Independent Variables. In equations in which y is expressed as a function of x, as in the above, the variable x is called the *independent variable* and y is called the *dependent variable*.

Geometric Illustrations. The area S and the circumference s of a circle are functions of the radius r:

$$S = \pi r^2 \quad \text{and} \quad s = 2\pi r.$$

The surface area S and the volume V of a sphere are functions of the radius r;

$$S = 4\pi r^2 \quad \text{and} \quad V = \tfrac{4}{3}\pi r^3.$$

One variable may be expressed as a function of two or more different variables. The lateral surface area S and the volume V of a right circular cylinder are functions of the radius r of the base and the altitude h;

$$S = 2\pi r h \quad \text{and} \quad V = \pi r^2 h.$$

The lateral surface area S of a right circular cone is a function of the radius r of the base and the slant height p, while the volume V is a function of the base and the altitude h;

$$S = \pi r p \quad \text{and} \quad V = \tfrac{1}{3}\pi r^2 h.$$

4. Functional Notation.

It is customary to express that y is a function of x by writing

$$y = f(x).$$

This symbolic expression is read "f of x."

If we write

$$f(x) = x^2 - 2x - 3,$$

x is the independent variable and $f(x)$ represents the dependent variable.

The notation for a function of x is useful in representing the value of the function for assigned values of the variable. This is done by replacing

the variable in the symbol by its value. Thus for

$$x = 2, \qquad f(2) = (2)^2 - 2(2) - 3 = 3,$$

$$x = a, \qquad f(a) = a^2 - 2a - 3$$

and $\qquad\qquad\qquad f(-x) = x^2 + 2x - 3.$

Exercise 1

GROUP A

Express each of the following functions algebraically.
1. The surface area of a cube is a function of its edge.
2. The volume of a rectangular parallelopiped with a square base is a function of its dimensions.
3. The area of a circle is a function of its circumference.
4. The volume of a sphere is a function of the area of a great circle.
5. The variable y is directly proportional to x. Find the proportionality factor if $x = 3$ when $y = 2$. Give the geometric interpretation of the equation.
6. The variable y is inversely proportional to x. Find the constant of proportionality if $x = 6$ when $y = 3$. Draw a graph of the function obtained.
7. If $f(x) = x^3 - x^2 + 2x - 3$, find $f(0), f(3), f(-2)$ and $f(-x)$.
8. If $f(x) = x^{2/3} - x^{-2/3}$, find $f(8), f(27)$ and $f(-8)$.
9. If $x^2 + xy - y^2 = 0$, find $y = f(x)$.
10. If $x^2 + xy + 2x - 2y + 3 = 0$, find $y = f(x)$.
11. If $f(x) = x^3 - x^2 - 2x + 2$, find $|f(-1)|$ and $|f(-2)|$.
12. If $x^2 + y^2 = 25$, find $y = f(x), f(3), |f(3)|$ and $|f(4)|$.

GROUP B

Express each of the following algebraically.
13. The volume of a sphere is a function of its diameter.
14. The surface area of a sphere is a function of its diameter.
15. The volume of a sphere is a function of its surface area.
16. The surface area of a sphere is a function of its volume.
17. The speed of a moving body varies directly as the square of the distance moved.
18. The intensity of light is inversely proportional to the square of the distance from the source.
19. If $f(x) = \sqrt{1 - x^2}$, find $f(0), f(1), f(\sin \theta)$ and $f(\cos \theta)$.
20. If $f(\theta) = \sin \theta$, show that $f(x + y) = \sin x \cos y + \sin y \cos x$.
21. If $f(\theta) = \cos \theta$, show that $f(x + y) = \cos x \cos y - \sin x \sin y$.
22. If $f(\theta) = \sin \theta$, show that $f(2x) = 2 \sin x \cos x$.
23. If $f(\theta) = \cos \theta$, show that $f(2y) = \cos^2 y - \sin^2 y$.
24. If $f(\theta) = \tan \theta$, show that $f(2A) = \dfrac{2 \tan A}{1 - \tan^2 A}$.

25. If $f(\theta) = \sin^2\theta$, show that $f\left(\dfrac{x}{2}\right) = \dfrac{1 - \cos x}{2}$.

26. If $f(\theta) = \cos^2\theta$, show that $f\left(\dfrac{x}{2}\right) = \dfrac{1 + \cos x}{2}$.

GROUP C

27. If the length of the hypotenuse of a right triangle is 8 inches, express the area as a function of one of the acute angles.

28. If a rectangle is inscribed in a circle of radius r, express the area of the rectangle as a function of one of its sides.

29. If a rectangle is inscribed in an isosceles triangle of base b and altitude a, express the area of the rectangle as a function of one of its sides.

30. If a right circular cone is inscribed in a sphere of radius r, express the volume of the cone as a function of the radius of its base.

31. If a right circular cylinder is inscribed in a sphere of radius r, express the volume of the cylinder as a function of the radius of its base

32. If a right circular cylinder is inscribed in a right circular cone whose altitude is a and the radius of whose base is b, express the volume of the cylinder as a function of the radius of its base.

33. If $f(x) = \sin x - \cos x$, find $f(0)$, $f(\pi/4)$, $f(\pi/2)$ and $f(\pi)$.

34. If $f(x) = 2^x$, find $f(2)$, $f(3)$, $f(0)$ and $f(-2)$.

35. If $f(t) = a^t$, show that $f(x + y) = a^x a^y$.

36. If $f(y) = 3^y$, show that $f(2x) = 9^x$.

37. If $f(x) = \log x$, show that $f(MN) = \log M + \log N$.

38. If $f(x) = \log x$, show that $f(y^2) = 2 \log y$.

39. If $f(x) = \log x$, show that $f(M/N) = \log M - \log N$.

40. If $f(x) = \log x$, show that $f(\sqrt{y}) = \tfrac{1}{2} \log y$.

5. Limits.

An idea which is essential to the study of the calculus is that of the limit of a variable. A variable which approaches a constant limit is encountered in the study of elementary geometry, where the area of a circle is defined as the limit approached by the area of a regular inscribed polygon as the number of sides is increased indefinitely.

A variable x is said to approach the limit a if $|x - a|$ becomes and remains less than any preassigned positive number ϵ, however small. To indicate that x approaches a as a limit, we write

$$x \to a \quad \text{or} \quad \lim x = a.$$

Hence, $x \to a$, if for each preassigned ϵ, however small but greater than zero, $|x - a| < \epsilon$.

In the application of the definition of the limit of a variable, two cases are to be considered. The first is the consideration of an independent

variable which may be made to approach some constant as a limit in an arbitrary manner. The second is a study of the behavior of a function as the independent variable approaches a limit.

As an example of a variable which approaches a limit, suppose that x takes on the following sequence of values:

$$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \cdots, \frac{n}{n+1}, \cdots.$$

We wish to prove that as x progressively assumes the values in the terms of this series, x approaches the limit 1.

Let us first choose a small positive number ϵ and then find a value of x in the series such that $|x - 1|$ is less than the number chosen. Since

$$|x - 1| = \left|\frac{n}{n+1} - 1\right| = \frac{1}{n+1},$$

we wish to show that it is possible to find values of n such that $1/(n+1) < \epsilon$, however small ϵ is chosen. This we do as follows:

If $\epsilon = \dfrac{1}{100}$, then $n = 100, 101, 102, \cdots$, give $\dfrac{1}{n+1} < \dfrac{1}{100}$.

If $\epsilon = \dfrac{1}{1000}$, then $n = 1000, 1001, \cdots$, give $\dfrac{1}{n+1} < \dfrac{1}{1000}$.

This process may be continued indefinitely by choosing a smaller and smaller number for ϵ. Thus it has been shown that x approaches the limit 1, since $|x - 1|$ becomes and remains less than any preassigned positive number, however small.

If x takes on the values given by the sequence

$$2, 4, 6, 8, 10, 12, \cdots, 2n, \cdots,$$

x does not approach a limit. In many cases it is possible to show that the limit of a variable *exists*, even though it may be difficult to find that limit.

Limit of a Function. If we have a function of the variable x, $f(x)$, it is frequently necessary to observe the behavior of $f(x)$ as x approaches a limit. As x approaches the limit a, if the absolute value of the difference between $f(x)$ and some constant L ultimately becomes and remains less than any preassigned positive constant, however small, the function $f(x)$ is said to *approach the limit L*. The symbol

$$\lim_{x \to a} f(x) = L,$$

is read " the limit of $f(x)$, as x approaches a, is L."

6. Theorems on Limits.

We shall desire to use the following theorems concerning limits. These theorems are given without proof. In each case the limit is assumed to exist.

Theorem I. *The limit of the algebraic sum of a finite number of variables is equal to the same algebraic sum of their limits:*

$$\lim_{x \to a} (u + v + w + \cdots) = \lim_{x \to a} u + \lim_{x \to a} v + \lim_{x \to a} w + \cdots,$$

in which u, v, w, $\cdots$ are assumed to be functions of x.

Theorem II. *The limit of the product of any finite number of variables is equal to the product of their respective limits:*

$$\lim_{x \to a} (u \cdot v \cdot w \cdots) = \lim_{x \to a} u \cdot \lim_{x \to a} v \cdot \lim_{x \to a} w \cdots.$$

Theorem III. *The limit of the quotient of two variables is equal to the quotient of their respective limits, provided that the limit of the denominator is not zero:*

$$\lim_{x \to a} \left(\frac{u}{v} \right) = \frac{\lim_{x \to a} u}{\lim_{x \to a} v}, \quad \lim_{x \to a} v \neq 0.$$

The statement of Theorem III assumes that the limit of the denominator is different from zero. Suppose that $\lim_{x \to a} v = 0$, two cases arise. First, if $\lim_{x \to a} u \neq 0$, then the fraction u/v may be made to take on values greater than any assigned constant by taking v sufficiently small. Consequently, the fraction cannot approach a limit. Second, if also $\lim_{x \to a} u = 0$, then the theorem does not apply, and the ratio of the limits takes the form $0/0$, which has no meaning. (However, the *limit of the ratio may exist* as we shall see in the following chapters.)

Consider the following functions for which we wish to find their limits as x approaches the limit 2.

$$f_1(x) = x^2 + 3x, \quad f_2(x) = \frac{3x + 2}{x^2}.$$

In application of Theorems I and II,

$$\lim_{x \to 2} f_1(x) = \lim_{x \to 2} x^2 + \lim_{x \to 2} 3x$$

$$= (\lim_{x \to 2} x)^2 + 3 \lim_{x \to 2} x = 10.$$

In application of Theorem III,

$$\lim_{x \to 2} f_2(x) = \frac{\lim_{x \to 2} (3x + 2)}{\lim_{x \to 2} x^2} = \frac{3 \lim_{x \to 2} x + 2}{(\lim_{x \to 2} x)^2} = 2.$$

As an illustration of the special case considered above, the

$$\lim_{x \to 2} \frac{x^2 - 4}{2x - 4} = \frac{0}{0}$$

exists, despite the fact that in its present form the limit appears to be meaningless. Since,

$$\frac{x^2 - 4}{2x - 4} = \frac{x + 2}{2}, \quad \lim_{x \to 2} \frac{x^2 - 4}{2x - 4} = \lim_{x \to 2} \frac{x + 2}{2} = 2.$$

7. Continuous Functions.

A function of one variable is said to be *continuous* if a sufficiently small change in the variable produces a small change in the value of the function. The continuity of a function is such an important concept that its precise meaning is stated with some care.

A function $f(x)$ is said to be continuous for $x = a$ if

$$\lim_{x \to a} f(x) = f(a).$$

A function is continuous in the interval $x_1 < x < x_2$, if it is continuous for all values of x within this interval, or *range*.

A function is said to be discontinuous for $x = a$, if the condition for continuity is not satisfied.

Infinity. The most important type of discontinuity with which we shall be concerned is that in which the function *increases without limit* as its variable approaches a limit. If the function $f(x)$ increases without limit, as x approaches a, we say that the function *becomes infinite*. In this case, we write

$$\lim_{x \to a} f(x) = \infty.$$

It is understood that the *symbol ∞ does not represent a number,* but that it is a symbolic representation of a variable increasing beyond any limit.

Geometrically, a discontinuity of a function occuring at $x = a$, means that the curve $y = f(x)$ approaches nearer and nearer to the line $x - a = 0$,

usually without reaching it, as the curve recedes farther and farther from the x-axis.

As an example of a function possessing a discontinuity, the function

$$f(x) = \frac{1}{x - 1}$$

is continuous for all values of x, save for $x = 1$ for which it is not defined. As x approaches 1 from values less than 1, $f(x)$ becomes *negatively* infinite. As x approaches 1 from values greater than 1, $f(x)$ becomes *positively* infinite.
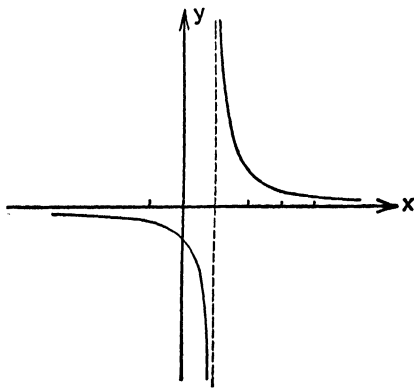
Fig. 1

The curve representing the given function, drawn with reference to the x-axis and the $f(x)$-axis, is the hyperbola in Figure 1. The curve is shown to be continuous for all values of x less than 1 and for all values of x greater than 1. At $x = 1$, the curve is discontinuous and possesses a vertical *asymptote* which is the line $x - 1 = 0$.

Exercise 2

GROUP A

If x assumes each of the following sequences of values, find the limit where it exists and prove that it is the limit by the use of an arbitrarily small positive quantity ϵ.

1. $\dfrac{1}{2}, \dfrac{3}{4}, \dfrac{5}{6}, \dfrac{7}{8}, \cdots \dfrac{2n - 1}{2n}, \cdots$.

2. $0.6, 0.66, 0.666, 0.6666, \cdots$.

3. $1, 2, 3, 4, 5, \cdots, n, \cdots$.

Find the limit of each of the following functions, where it exists, using the theorems of the last section.

4. $\lim\limits_{x \to 1} (x^3 - 2x^2 + x - 2)$.

5. $\lim\limits_{x \to 2} \dfrac{x^2 - x - 2}{x + 2}$.

6. $\lim\limits_{x \to -1} \dfrac{x^2}{x + 1}$.

7. $\lim\limits_{x \to 3} (x - 3)(x + 4)$.

8. If $\lim x = a$ and $\lim y = 0$, prove that $\lim(xy) = 0$.

9. Prove that $\lim x^2 = (\lim x)^2$.

10. Prove that $\lim 4x = 4 \lim x$.

GROUP B

If x assumes each of the following sequences of values, find the limit where it exists and prove that it is the limit by the use of an arbitrarily small positive quantity ϵ.

11. $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \cdots$.

12. $1, 2, 1, 3, 1, 4, 1, 5, \cdots$.

13. $\frac{1}{2}, \frac{-2}{3}, \frac{3}{4}, \frac{-4}{5}, \frac{5}{6}, \frac{-6}{7}, \cdots$.

Find the limit of each of the following functions, where it exists, using the theorems of the last section.

14. $\operatorname*{Lim}_{x \to 0} \dfrac{x+2}{x-3}$.

15. $\operatorname*{Lim}_{x \to \infty} \dfrac{x-1}{2x+1}$.

16. $\operatorname*{Lim}_{x \to \pi} \sin x$ and $\operatorname*{lim}_{x \to \pi} \cos x$.

17. $\operatorname*{Lim}_{x \to \pi/2} \tan x$ and $\operatorname*{lim}_{x \to \pi} \cot x$.

18. $\operatorname*{Lim}_{x \to 0} \dfrac{\sin x}{\tan x}$ and $\operatorname*{lim}_{x \to 0} \dfrac{\cot x}{\cos x}$.

19. Prove that $\lim \left(\dfrac{x}{y}\right) = \dfrac{\lim x}{\lim y}$, provided that $\lim y \neq 0$.

20. Prove that $\lim (xy) = \lim x \cdot \lim y$.

GROUP C

21. Find $\operatorname*{lim}_{x \to 0} \dfrac{\sin 2x}{\sin x}$

22. Show that $\operatorname*{lim}_{x \to 0} \dfrac{a^x - a^{-x}}{a^x + a^{-x}} = 0$

23. If $f(x) = x^2$, find $\operatorname*{lim}_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

24. If $f(x) = \dfrac{1}{x}$, find $\operatorname*{lim}_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

25. If $f(x) = x - x^3$, find $\operatorname*{lim}_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

26. Show that $\operatorname*{lim}_{x \to -2} \dfrac{x^2 - 4}{x^2 + 4x + 4}$ does not exist.

27. Show that $\operatorname*{lim}_{x \to 0} \sin (1/x)$ does not exist.

28. Show that $\operatorname*{lim}_{x \to 0} \dfrac{1}{\cos x}$ exists and that $\operatorname*{lim}_{x \to \pi/2} \dfrac{1}{\cos x}$ does not.

29. Show that $\operatorname*{lim}_{x \to \pi/2} \tan x$ does not exist and that $\operatorname*{lim}_{x \to \pi/4} \dfrac{x}{\tan x}$ does.

8. Polynomials.

A *rational integral function* or, a *polynomial*, is defined by

$$f(x) = b_0 x^n + b_1 x^{n-1} + b_2 x^{n-2} + b_3 x^{n-3} + \cdots + b_n,$$

The irrational roots of an equation, if any, may be approximated by several methods. However, if no great accuracy is desired, an irrational root may be approximated by narrowing the interval in which the root lies. Since the curve $y = f(x)$ is a continuous curve, the equation $f(x) = 0$ has at least one root between x_1 and x_2 if $f(x_1)$ and $f(x_2)$ have opposite signs.

The equation

$$f(x) = x^3 + x - 6 = 0$$

has a real irrational root between 1 and 2 since

$$f(1) = -4 \quad \text{and} \quad f(2) = 4.$$

Moreover, this root lies between 1.6 and 1.7 since

$$f(1.6) = -0.304 \quad \text{and} \quad f(1.7) = 0.613.$$

Tedious though this process is, it can be continued to give any desired degree of accuracy.

Exercise 3

GROUP A

1. Show that $f(x) = x^2$ is a continuous function for all values of x.
2. For what values of x is $f(x) = \dfrac{2}{x-2}$ continuous? For what value is it discontinuous? Draw the curve.
3. Show that the function $f(x) = x + 2/x$ has a discontinuity. Draw the curve $y = f(x)$.
4. Illustrate the remainder theorem using the function $f(x) = x^3 - x^2 + x + 6$ and $a = 4$.
5. Draw the graph of $f(x) = x^3 - 4x^2 + x + 5$.
6. Draw the graph of $f(x) = x^4 - 13x^2 + 20$ and show that it is continuous at every point.
7. For what value of x is $f(x) = \dfrac{x-1}{x-2}$ discontinuous? Draw the curve.

Find all the roots of each of the following equations.

8. $x^3 - 4x^2 + 4x - 3 = 0$.
9. $x^3 - 7x - 6 = 0$.
10. $2x^3 - x^2 + x + 1 = 0$.

GROUP B

11. Show that $y = \sin x$ is a continuous function for all values of x.
12. For what values of x is $f(x) = \dfrac{x+1}{x^2-9}$ continuous. For what values is it discontinuous? Draw the curve.

13. Show that $f(x) = x + 8/x^2$ has a discontinuity.   Draw the curve.
14. Find the value of B so that 3 is a root of $x^3 - 2x^2 + Bx - 3 = 0$.
15. For what real value of x is $f(x) = 2x^3 - 3x + 4$ equal to 14?
16. Find the values of B and C so that -2 and 1 are roots of $x^3 - 2x^2 + Bx + C = 0$.
17. For what values of x do the functions $f_1(x) = 4x^2 - 6x + 3$ and $f_2(x) = x^3 - 2x^2 + 5x - 3$ have equal values?
18. Find the value of k so that a chord perpendicular to the axis of the parabola $y = kx^2$ may be 6 inches long and may be 8 inches from the vertex.
19. Locate between consecutive integers the real roots of $x^3 + 3x^2 - 2x - 5 = 0$.
20. Approximate the real irrational root of $x^3 - 3x^2 + 4x - 6 = 0$ to one decimal place.

GROUP C

21. Draw the curves $y = 2^x$ and $y = 2^{-x}$.
22. Draw the curve $y = 2^{1/x}$.
23. For what values of x is $f(x) = \tan 2x$ discontinuous?
24. Discuss the continuity of $y = \dfrac{1}{1 - \cos x}$.
25. Given $y = f(x)$, a polynomial. Is y^2 a continuous function? Is $1/y$ a continuous function?
26. Given the two continuous functions $y = f_1(x)$ and $y = f_2(x)$. Consider the continuity of $f_1(x) + f_2(x)$, $f_1(x) \cdot f_2(x)$ and $\dfrac{f_1(x)}{f_2(x)}$.
27. Prove the remainder theorem.
28. Prove that imaginary roots of a quadratic equation having real coefficients are conjugate imaginaries.
29. Prove that imaginary roots of any rational integral equation having real coefficients, occur in conjugate pairs.
30. Show that the constant term of an equation is plus or minus the product of all the roots, provided that the coefficient of the highest power of x is 1.

CHAPTER II

DIFFERENTIATION OF POLYNOMIALS

10. Increments.

The differential calculus may be said to be primarily concerned with the problem of finding how a function varies in comparison with the independent variable. In general, a change in the value of the variable will produce a change in a function of that variable. It is from the study of a comparison of these changes that the fundamental operation known as differentiation is developed.

An arbitrary change in an independent variable is called an *increment* of that variable. For convenience, this increment is ordinarily taken to be positive. Corresponding to any increment of the independent variable, in general, there is produced an increment of the function which may be positive, negative or zero.

If $y = f(x)$, the increments of the variables x and y are denoted by the symbols

$$\Delta x \quad \text{and} \quad \Delta y,$$

respectively, which are read " delta x " and " delta y." It is to be understood that these symbols do not represent a product of Δ and each of the variables, but that Δx represents an *arbitrary* change in x and that Δy represents the *corresponding* change produced in the function y.

In the function $y = f(x)$, if x is given an increment, the final value of the variable is $x + \Delta x$. Hence, the final value of the function is

$$y + \Delta y = f(x + \Delta x).$$

From the given value of the function and this one, the increment of the function is obtained by subtraction, giving

$$\Delta y = f(x + \Delta x) - f(x).$$

Let the area of a square whose variable side is x be denoted by the variable y. Then

$$y = x^2.$$

If the side x of the square is given the increment Δx, the corresponding increment of the area y is Δy. Then

$$y + \Delta y = (x + \Delta x)^2.$$

15

From these two equations the increment of the area is obtained,

$$\Delta y = 2x\,\Delta x + \overline{\Delta x}^2.$$

The square of area x^2 is drawn in Figure 3. When a side is increased by a length Δx, it is seen that the area is increased by two rectangles each of area $x\,\Delta x$ and the square of area $\overline{\Delta x}^2$.
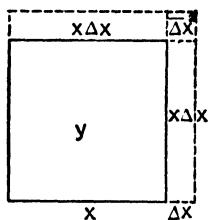
Let us consider the function

$$s = 16t^2,$$

in which s represents the distance in feet from the starting point of a body falling in a vacuum after t seconds and in which the acceleration due to gravity is taken to be 32 feet per second per second.

Fɪɢ. 3

Let the distance which the body has fallen in the first t seconds be represented by OA in Figure 4. Also, let the distance which the body has fallen in the first $t + \Delta t$ seconds be represented by OB. Then, for a particular value of t,

$$OA = s = 16t^2$$

$$OB = s + \Delta s = 16(t + \Delta t)^2$$

and

$$AB = \Delta s = 16(2t\,\Delta t + \overline{\Delta t}^2).$$

In both of the above illustrations, it is to be observed that the increment of a function is expressed in terms of both the independent variable and its increment.

11. Average Rate of Change.

Consider again the problem of the freely falling body, where

$$s = 16t^2.$$

Fɪɢ. 4

In the last section it was found that the distance fallen in the interval of time from t to $t + \Delta t$, that is, during the time Δt is

$$\Delta s = 32t\,\Delta t + 16\overline{\Delta t}^2.$$

The ratio of the change of distance to the change of time,

$$\frac{\Delta s}{\Delta t} = 32t + 16\Delta t,$$

is the *average speed* of the body during the time-interval Δt.
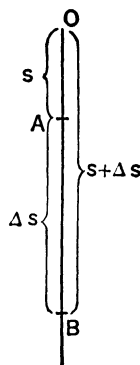
For example, in this problem the average speed from $t = 2$ to $t = 2.1$ seconds is found as follows: Since $t = 2$ and $\Delta t = 0.1$,

$$\frac{\Delta s}{\Delta t} = 32(2) + 16(0.1) = 65.6 \text{ ft./sec.}$$

This average speed represents the average change of distance per *unit* of time, or the average rate of change of s with respect to t in the interval Δt. Our result means that if the falling body moved at a uniform speed of 65.6 ft./sec. for 0.1 sec. after the end of the second second, it would have traversed a distance of 6.56 feet.

As a second illustration of average rate of change, let us return to the problem of the square, where

$$y = x^2.$$

In the last section it was found that the increment of the area corresponding to an increment of the side is

$$\Delta y = 2x \, \Delta x + \overline{\Delta x}^2.$$

The ratio of the change of area to the change of side,

$$\frac{\Delta y}{\Delta x} = 2x + \Delta x,$$

is the average rate of change of the area with respect to the side in the interval from x to $x + \Delta x$.

If we wish to find the average rate of change of area per unit side as x increases from 3 to 3.2 inches, we proceed as follows: Since $x = 3$ and $\Delta x = 0.2$,

$$\frac{\Delta y}{\Delta x} = 2(3) + (0.2) = 6.2 \text{ sq. ins. per in.}$$

As in the previous sections, let y represent any polynomial, given by the equation

$$y = f(x),$$

which defines the value of y corresponding to any initial value of x. If x is given an increment, the increment of the function was found to be

$$\Delta y = f(x + \Delta x) - f(x).$$

Dividing by the increment of the independent variable,

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

This ratio is the *average rate of change* of the function with respect to x in the interval from x to $x + \Delta x$.

12. Rate of Change.

In the preceding section the average rate of change of a function over a given interval of the independent variable was defined and interpreted. This is an essential concept as it leads to the important definition of the *rate of change* of a function for a given value of the variable.

The *rate of change of a function with respect to the independent variable* is defined as *the limit of the ratio of the increment of the function to the increment of the variable as both increments approach zero.* Thus, from the preceding section,

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x},$$

provided that such a limit exists.

Once the initial value of x is chosen, it is considered to be fixed. Hence, the ratio of the increments may be regarded as a function of Δx alone which may or may not approach a limit as Δx approaches zero.

Let us find the rate of change of the function

$$y = x^2 - 2x$$

for $x = 3$.

The change in the function for any change of the variable is

$$\Delta y = 2x \, \Delta x + \overline{\Delta x}^2 - 2\Delta x.$$

The average rate of change of the function over any interval Δx is

$$\frac{\Delta y}{\Delta x} = 2x + \Delta x - 2.$$

The rate of change of the function for any value of x is

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} (2x + \Delta x - 2) = 2x - 2.$$

If $x = 3$, $\lim_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x} = 2(3) - 2 = 4$ units per unit of x.

Instantaneous Speed. An important physical application of the rate of change of a function with respect to its variable can be made when the distance of a particle moving in a straight line from a fixed point of that line is expressed as a function of the time,

$$s = f(t).$$

The *instantaneous speed* of the particle is the limit of the ratio of the displacement Δs and the time-interval Δt as Δt approaches zero:

$$\text{Speed} = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t}.$$

In the case of the freely falling body,

$$s = 16t^2, \quad \Delta s = 32t\,\Delta t + 16\overline{\Delta t}^2$$

and

$$\frac{\Delta s}{\Delta t} = 32t + 16\Delta t.$$

Hence,

$$\text{Speed} = \lim_{\Delta t \to 0} (32t + 16\Delta t) = 32t \text{ ft./sec.}$$

By means of the definition of the limit of a variable given in Section 5, it can be proved that the limit of the average speed of a falling body exists and has the value given above. It is to be recalled that, while t may have any value, once it is chosen it is fixed and, consequently, it behaves as a constant. However, the expression $(32t + 16\Delta t)$ is a variable which depends on the variable Δt. In order to prove that the limit of the former variable is $32t$ as Δt approaches zero, it must be shown that the absolute value of the difference of the two may be made to become and remain less than any small preassigned positive quantity ϵ, however small. Thus,

$$| (32t + 16\Delta t) - 32t | = | 16\Delta t |.$$

As yet, no value has been assigned to Δt. This is done *after* ϵ has been chosen. If we choose Δt so that

$$16\Delta t < \epsilon, \quad \text{or} \quad \Delta t < \frac{\epsilon}{16},$$

then the absolute value of the difference is smaller than ϵ, regardless of how small ϵ is taken. Hence, the proof is complete.

Exercise 4

GROUP A

In each of the following problems s represents the distance of a particle moving on a line from a fixed point of that line at any time t.

1. $s = 2t^2 - t + 3$. Find Δs and $\Delta s/\Delta t$.

2. $s = 2t^2 + t - 5$. Find the change in s corresponding to any change in t and corresponding to the change from $t = 2$ to $t = 2.1$.

3. $s = 1 + t - 3t^2$. Find the average speed during the interval of time Δt and from $t = 1$ to $t = 1.3$.

4. $s = t^3 + 2t + 1$.   Find $\Delta s/\Delta t$ and $\lim\limits_{\Delta t \to 0} \Delta s/\Delta t$.

5. $s = t^3 + 2t + 3$.   Find the speed at the end of 2 seconds.

6. Find the change in the volume of a cube of edge x for any change in the edge. Find the average rate of change of the volume with respect to the edge as the edge is increased from 2 ins. to 2 4 ins.

7. Given $y = 2x^2 - x + 1$.   Find Δy, $\Delta y/\Delta x$ and $\lim\limits_{\Delta x \to 0} \Delta y/\Delta x$.

8. Given $y = x^3 + x^2$.   Find the change in y and the average rate of change of y with respect to x for any change in x and as x changes from 1 to 1.2.

9. Given $y = 4x - x^2$.   Find the rate of change of y with respect to x for $x = 1$, $x = 2$ and $x = 3$.

10. A ball is thrown upward so that its distance from the ground at any time is given by $s = 480t - 16t^2$.   How long is it in the air?   With what speed is it thrown? With what speed does it strike the ground?   At what time is the speed zero? How high does it rise?

GROUP B

11. Given $y = x^3 - x^2 + x + 1$.   Find Δy, $\Delta y/\Delta x$ and $\lim\limits_{\Delta x \to 0} \Delta y/\Delta x$.

12. Given $y = x^3 + x^2 + x + 2$.   Find the change of the function and the average rate of change of the function with respect to x as x changes from 1.0 to 1.01.

13. Given $y = 1 + 3x - x^3$.   Find the rate of change of the function with respect to x for any value of x and for $x = 5$.

14. Find the rate of change of the volume of a cube with respect to its edge if the edge is 6 ins.

15. Find the rate of change of the circumference of a circle with respect to its radius.

16. Find the rate of change of the area of a circle with respect to its diameter.

17. Show that the rate of change of the area of a circle with respect to its radius is equal to its circumference.

18. Find the rate of change of the volume of a spherical balloon with respect to its radius.

19. Find the rate of change of the area of a sector of a circle of radius a with respect to the central angle.

20. The altitude of a right circular cone is always equal to the diameter of its base. Assuming the cone to expand, retaining its form and proportions, find the rate of change of the volume with respect to the radius of the base.

13. The Derivative.

The derivative of a function with respect to its variable is defined as *the limit of the ratio of the increment of the function to the increment of the variable as both increments approach zero.*   If $y = f(x)$, *the derivative of y with respect to x is*

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \quad \text{or,} \quad \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

The rate of change of a function with respect to the independent variable for a value of that variable is the numerical value of the derivative.

Symbols which may be used to represent the derivative of the function $y = f(x)$ with respect to x are

$$\frac{dy}{dx} \quad \text{and} \quad f'(x).$$

Similarly, the symbols to represent the derivative of the function $s = f(t)$ with respect to t may be

$$\frac{ds}{dt} \quad \text{and} \quad f'(t).$$

The process of finding the derivative of a function is called *differentiation*. To differentiate a function, give an increment to the independent variable, find the corresponding increment of the function and evaluate the limit of the quotient of these increments as both increments approach zero. This process is indicated in the definition and may be called the *delta process*.

The operational symbol for differentiation is d/dx. If functions of x are represented by the variables y, u or $f(x)$, the differentiation to be performed is often denoted by

$$\frac{d}{dx}y, \quad \frac{d}{dx}u, \quad \text{or} \quad \frac{d}{dx}f(x),$$

respectively. The value of these symbols lies in the fact that they indicate an operation or a procedure to be carried out with respect to a particular variable. We shall see later that differentiation with respect to different variables is an important variation of the process.

Some authors use D_x as the symbol to indicate differentiation with respect to x. Following this notation, the derivatives of the functions above are represented by

$$D_x y, \quad D_x u \quad \text{or} \quad D_x f(x).$$

Exercise 5

GROUP A

Differentiate each of the following functions using the delta process.

1. $y = 6x + x^3$.

2. $y = 2x + x^4$.

3. $y = x^3 - 2x^2 + x - 1$.

4. $y = (x - 1)^2$.

5. $y = \dfrac{1}{x}$.

6. $y = \dfrac{1}{x + 1}$.

7. Find the coordinates of the point on the curve $y = x^2 - 6x$ at which the rate of change of y with respect to x is zero.
8. Find the coordinates of the point on the curve $y = x^2 - 6x + 10$ at which the rate of change of y with respect to x is equal to 2.
9. A circular metal plate is heated and expands, remaining circular. Find the rate of change of the area with respect to the circumference when the radius is 5 ins.
10. An expanding balloon remains spherical. Find the rate of change of the surface with respect to the diameter when the radius is 10 ins.

GROUP B

Differentiate each of the following functions using the definition of the derivative.

11. $y = x - \dfrac{2}{x}$.

12. $y = \dfrac{1}{1 - x}$.

13. $y = x^2 + \dfrac{3}{x}$.

14. $y = \dfrac{1}{x^2}$.

15. $y = x^4 - x$.

16. $y = \dfrac{x + 1}{x - 1}$.

17. Find the rate of change of the area of an equilateral triangle with respect to its side.
18. Find the rate of change of the sector of a circle with respect to the radius if the central angle is $\pi/3$ radians.
19. Find the values of x for which the rates of change of the two given functions are equal

$$f_1(x) = x^3 - 3x^2 + 6x + 1, \quad f_2(x) = 2x^3 + 3x^2 - 30x + 8.$$

20. Find the coordinates of the points on the curves $y = 1/x$ and $y = x - x^2$ at which the rates of change of y with respect to x are equal.

GROUP C

Differentiate each of the following:

21. $y = \dfrac{x}{x + 1}$.

22. $y = \dfrac{1}{x^2 + 1}$.

23. $y = \sqrt{x}$.

24. $y = \dfrac{1}{\sqrt{x}}$

25. $y = \sqrt{x - 1}$.

26. $y = \sqrt{x^2 + 4}$.

27. Find the rate of change of the total surface area of a right circular cone with respect to the radius of its base, if the slant height remains 5 ft.
28. Find the rate of change of the lateral surface area of a right circular cone with respect to its altitude, if the radius of the base remains 4 ft.
29. The volume of a gas varies inversely as the pressure. When the pressure is 10 lbs. per sq. in., the volume is 500 cu. ins. Find the rate of change of the volume with respect to the pressure when the pressure is 100 lbs. per sq. in. Interpret the result.
30. The intensity of light on a surface varies inversely as the square of the distance between the surface and the source of the light. If the intensity is 500 units when the distance is 1 ft., find the rate of change of the intensity with respect to the distance when the distance is 10 ft.

14. Differentiation Formulas.

In this section *standard formulas* for the differentiation of simple algebraic functions are derived. The use of these formulas effects a saving of time, obviating the necessity of the evaluation of a special limit in each problem.

Derivative of a Constant. *The derivative of a constant is zero.*

Let
$$y = a,$$

where a is any constant. Then

$$y + \Delta y = a \quad \text{and} \quad \Delta y = 0.$$

Hence,
$$\frac{\Delta y}{\Delta x} = 0 \quad \text{and} \quad \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = 0.$$

(1)
$$\frac{da}{dx} = 0.$$

Derivative of x to a Positive Integral Power. *The derivative of x^n is nx^{n-1}.*

Let
$$y = x^n,$$

where n is any positive integer. Then

$$y + \Delta y = (x + \Delta x)^n.$$

The right-hand member may be expanded by the binomial theorem, giving

$$y + \Delta y = x^n + nx^{n-1}\,\Delta x + \frac{n(n-1)}{1\cdot 2}\,x^{n-2}\,\overline{\Delta x}^2 + \cdots + \overline{\Delta x}^n.$$

Subtracting the first equation from this one, we have

$$\Delta y = nx^{n-1}\,\Delta x + \frac{n(n-1)}{1\cdot 2}\,x^{n-2}\,\overline{\Delta x}^2 + \cdots + \overline{\Delta x}^n.$$

Dividing both sides of the equation by Δx,

$$\frac{\Delta y}{\Delta x} = nx^{n-1} + \frac{n(n-1)}{1\cdot 2}\,x^{n-2}\,\Delta x + \cdots + \overline{\Delta x}^{n-1}.$$

Taking the limit as Δx approaches zero,

(2)
$$\frac{d(x^n)}{dx} = nx^{n-1}$$

Derivative of a Constant times a Function. *The derivative of constant times a function is the constant times the derivative of the function.*

Let
$$y = au,$$

where a is any constant and u is a function of x which can be differentiated. If x is given the increment Δx, u and y will have corresponding increments Δu and Δy, respectively. Then

$$y + \Delta y = a(u + \Delta u) \quad \text{and} \quad \Delta y = a\,\Delta u.$$

Dividing both sides of the equation by Δx,

$$\frac{\Delta y}{\Delta x} = a\,\frac{\Delta u}{\Delta x}, \quad \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = a \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x}.$$

In taking the limit as Δx approaches zero, Δu also approaches zero.

$$(3) \qquad\qquad \frac{d(au)}{dx} = a\,\frac{du}{dx}.$$

Derivative of a Sum. *The derivative of the algebraic sum of two or more functions is the same algebraic sum of their derivatives.*

Let $$y = u + v,$$

where u and v are functions of x which can be differentiated. If x is given the increment Δx, u, v and y will have the corresponding increments Δu, Δv and Δy, respectively. Then

$$y + \Delta y = (u + \Delta u) + (v + \Delta v), \quad \Delta y = \Delta u + \Delta v,$$

$$\frac{\Delta y}{\Delta x} = \frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x}, \quad \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} + \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x}.$$

In taking the limit as Δx approaches zero, Δu and Δv also approach zero.

$$(4) \qquad\qquad \frac{d(u + v)}{dx} = \frac{du}{dx} + \frac{dv}{dx}.$$

Although the derivation given for formula (2) is valid for positive integral values of n only, the formula holds for positive and negative integral and rational fractional values of n. The general proof is to be found in Section 35.

Exercise 6

GROUP A

Differentiate each of the following functions.

1. $y = 2x^3 - 3x^2 + 6x + 9.$
2. $y = 3x^4 - 2x^3 + x - 4.$
3. $f(x) = 1 - 2x - 3x^2 - x^3.$
4. $f(x) = 2 - x - 2x^3 - 3x^5$

5. $y = 1/x$. Apply the delta process and compare with the result by use of formula (2).

6. $y = \sqrt{x}$. Apply the delta process and compare with the result obtained by the use of formula (2).

7. $f(x) = 2\sqrt{x^3} - 3\sqrt{x} + 8$.

8. $f(x) = \dfrac{2}{x} + \dfrac{6}{x^3} - \dfrac{1}{x^2}$.

9. $f(x) = 4x^{-2} - 6\sqrt{x} + \dfrac{2}{\sqrt{x}}$.

10. $y = \dfrac{x^2}{4} - \dfrac{5x}{4} + \dfrac{3}{2} - \dfrac{3}{x} - \dfrac{2}{x^2}$.

GROUP B

Differentiate each of the following functions.

11. $f(x) = (x^2 - 1)^2$.

12. $f(x) = x(x + 2)^3$.

13. $f(x) = x^2(\sqrt{x} - x^{-1})$.

14. $y = kx(ax^2 + bx + c)$.

15. $y = \dfrac{3x^2 - 2x^3}{\sqrt{x}}$.

16. $y = ax\left(\sqrt{x} + \dfrac{b}{\sqrt{x}}\right)$.

In each of the following problems the distance s of a particle moving on a line from a fixed point of that line is given as a function of the time t.

17. $s = 2t^3 - 15t^2 + 48t + 24$. Find the times at which the speed of the particle is 12 units per unit of time.

18. $s = t^4 - 12t^3 + 46t^2 - 60t + 2$. Where does the particle start motion? Find the times at which the speed is zero.

19. $s = t^4 - 8t^3 + 22t^2 - 24t + 8$. Find the times at which the particle comes to rest.

20. $s = t^4 - 6t^3 + 11t^2 - 6t$. Find the times at which the particle passes the fixed point. Approximate to one decimal place each of the times at which the particle comes to rest..

15. Geometric Interpretation of the Derivative.

A straight line which is determined by two separate points of a curve is a *secant* of the curve. If one of the two points is fixed while the second point is allowed to move along the curve with the first point as its limiting position, the secant will rotate about the fixed point and ordinarily will approach a limiting position. A *tangent* to a curve at a fixed point of the curve is defined as the *limiting position of a secant through the fixed point and a moving point as the latter point approaches the fixed point as a limiting position.*

The coordinates of any point P on the curve $y = f(x)$ are (x, y). Let the coordinates of any other near point P' be $(x + \Delta x, y + \Delta y)$. In Figure 5 the directed line segments PR and RP' are Δx and Δy, respectively. While the point P is any point of the curve, once it is chosen, it is considered to be a fixed point.

<center>Fɪɢ. 5</center>
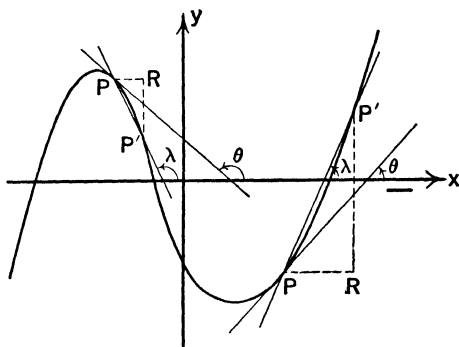
Let λ represent the inclination of the secant PP' for any position of P', then

$$\tan \lambda = \frac{\Delta y}{\Delta x}.$$

Hence, *the slope of the secant PP' represents the average rate of change of a function with respect to the independent variable over the interval from x to $x + \Delta x$.*

As the point P' approaches the fixed point P, Δx approaches zero. Let θ represent the inclination of the tangent PQ to the curve at the point P. Then

$$\lim_{\Delta x \to 0} \lambda = \theta, \quad \text{and} \quad \lim_{\Delta x \to 0} \tan \lambda = \tan \theta.$$

Hence,

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \tan \theta, \quad \text{or}$$

$$f'(x) = \frac{dy}{dx} = \tan \theta.$$

In words, *the slope of the tangent to a curve $y = f(x)$ at any point P is the numerical value of the derivative $f'(x)$ obtained by evaluating it for the abscissa of P.*

The slope of the tangent to a curve at a given point is called the *slope of the curve* at that point. Hence, *the slope of a curve $y = f(x)$ at any given point represents the rate of change of the function $f(x)$ with respect to the independent variable for the abscissa of the given point.*

It should be remarked that the derivative of a function is not another name for the slope of a curve. Quite the contrary, the use of the derivative for finding the slope of a curve from its equation is but one of the many applications of the derivative of a function. However, the geometric interpretation of the derivative of a function is often useful in studying the derivative and its meaning in other applications.

16. Equations of Tangents to Curves.

It is now possible to write the equation of the tangent to a curve $y = f(x)$ at any one of its given points, since the slope of the tangent can be found from the derivative. If the abscissa of the given point is x_1, the slope of the tangent to the curve at this point is the numerical value obtained from the derivative for $x = x_1$. This value is indicated by the symbol

$$f'(x_1).$$

Hence, the equation of the tangent to the curve at the given point is

$$y - f(x_1) = f'(x_1)(x - x_1).$$

Let it be required to write the equation of the tangent to the curve $f(x) = x^3 - 3x^2 + 4x + 1$ at the point whose abscissa is 2.

From the derivative,

$$f'(x) = 3x^2 - 6x + 4, \quad f'(2) = 4.$$

From the given function, $f(2) = 5$. Hence, the equation of the tangent is

$$y - 5 = 4(x - 2),$$

or
$$4x - y - 3 = 0.$$

Exercise 7

GROUP A

Find the equation of the tangent to each of the following curves at the indicated points.

1. $f(x) = x^2 - 6x + 10$ at $x = 4$.
2. $f(x) = x^3 - 2x^2 + x - 5$ at $x = 2$.
3. $f(x) = x^3 - x$ at $x = 2$.
4. $f(x) = 4 + 2x - x^2$ at $x = -1$.
5. $f(x) = x - (3/x)$ at $x = 3$.

6. Find the slope of the curve $y = 6x - x^2$ at each of the points $x = 2$, $x = 3$ and $x = 4$. Draw the curve and the tangents at the given points.

7. Find the slopes of the curve $y = x^2 - 4x - 1$ at $y = 4$. Draw the curve and the tangents at the given points.

8. Find the coordinates of the points at which the slope of the curve
$$f(x) = x^3 - 3x^2 - 8x + 7 \text{ is } 1.$$

9. Write the equation of the line perpendicular to the tangent to the curve $y = x^3 - 4x^2$ at $x = 1$ and passing through the point of tangency.

10. Given $f(x) = x^2 + 2x - 5$. Find the average rate of change of the function with respect to x from $x = 2$ to $x = 3$. Show that it is equal to the slope of the secant through the points on the curve whose abscissas are 2 and 3.

GROUP B

11. Find the coordinates of the points on the curve $y = 3x^4 - 2x^3 - 6x^2 + 6x - 9$ at which the rate of change of y with respect to x is zero. Write the equations of the tangents to the curve at these points.

12. Find the equations of the horizontal tangents to the curve $y = 3x^5 - 25x^3 + 60x + 3$.

13. Find the angle between the line $x - 3y + 1 = 0$ and the tangent to the curve $y = x^2 - 2x + 1$ at one of the points of intersection of the curve and the given line.

14. Given the line $4x - y + 5 = 0$ and the curve $y = x^2 + 2x + 6$. Is the line tangent to the curve? If so, write the equation of the perpendicular to it through the point of tangency.

15. A secant of the curve $x^2 - y - 2 = 0$ has a slope 2 and intersects the curve at the point $(-4,14)$. Find the angle between the curve and the secant at the second point of intersection of the curve and the secant.

16. Write the equations of the tangents to the curve $f(x) = x^3 + 4x^2 - 2x + 1$ which are parallel to the line $x - y + 8 = 0$.

17. Find the equations of the tangents to $y = x^3 + 3x^2 - 9x - 10$ which are perpendicular to the line $x + 15y - 23 = 0$.

18. Find the equation of the tangent to the curve $x^3 - 8y = 0$ at $x = 1$. .. Find the coordinates of the second point of intersection of the curve and the tangent.

19. Find the equation of the tangent to the curve $xy - 3y - x = 0$ at $x = 4$.

20. Find the rate of change of the slope of the curve $y = x^4 - x^3 - 3x^2 - 6x + 2$ at $x = -1$, at $x = 0$ and at $x = 1$.

17. Sign of the Derivative.

The general analytic meaning of the derivative of a function has been given as the measure of the rate of change of the function with respect to the independent variable. In addition, the geometric interpretation has been given as the measure of the slope of the curve. The consequences of these results are important and lead to significant applications.

For convenience, if $y = f(x)$, Δx was chosen to represent an arbitrary positive increment of x and Δy was chosen to represent the corresponding change of the function. Since the change of the function may be positive, negative or zero, the difference quotient $\Delta y/\Delta x$, has the same possibilities.

Also, since $\lim\limits_{\Delta x \to 0} \Delta y / \Delta x = f'(x)$, the numerical value of the derivative for $x = x_1$ may be positive, negative or zero. We consider the three cases as follows:
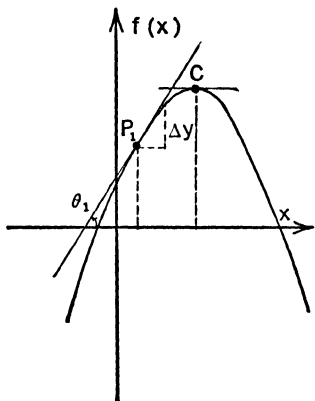
Fig. 6

If for $x = x_1$, the derivative of $f(x)$ is positive, that is,

$$f'(x_1) > 0,$$

the function is on the increase. Let P_1 be a point on the curve in Figure 6 at which the slope is positive. The inclination θ_1 of the tangent to the curve at this point is an acute angle. Hence, the curve necessarily rises from left to right. Thus, when the slope is positive, the function increases as x increases. A function is said to be an *increasing function* for those values of the variable for which its derivative is positive.

If for $x = x_2$, the derivative of $f(x)$ is negative, that is,

$$f'(x_2) < 0,$$

the function is on the decrease. Let P_2 be a point on the curve in Figure 7 at which the slope is negative. The inclination θ_2 of the tangent to the curve at this point is an obtuse angle. Hence, the curve necessarily falls from left to right. Thus, when the slope is negative, the function decreases as x increases. A function is said to be a *decreasing function* for those values of the variable for which its derivative is negative.

If for $x = c$, the derivative is zero, that is,

$$f'(c) = 0,$$

the function is neither on the increase nor on the decrease. Let C be a point on the curve in Figure 6 and D be a point on the curve in Figure 7 at each of which points, the slope is zero. The tangents to the curves at these points are parallel to the x-axis. A function is said to have **critical values** for those values of the variable for which the derivative is zero.
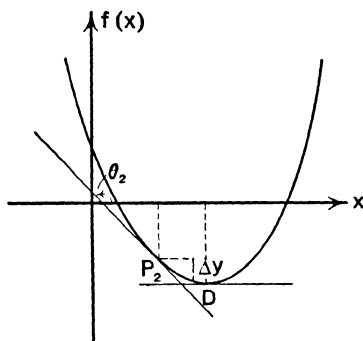
Fig. 7

The values of x for which a function $f(x)$ is critical, are the real solutions

of the equation $f'(x) = 0$. If the real roots are

$$x = c_1, \ c_2, \ c_3, \cdots, \ c_n,$$

then the critical values of the function are

$$f(c_1), \ f(c_2), \ f(c_3), \cdots, \ f(c_n).$$
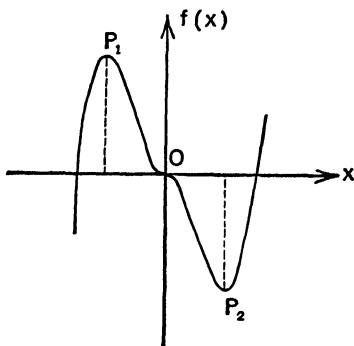
FIG. 8

To find the range of values of x for which a polynomial is increasing or decreasing, the derivative is often written in its factored form,

$$f'(x) = a(x - c_1)(x - c_2) \cdots (x - c_n),$$

so that we may find the sign of the derivative by considering the sign of each factor.

Consider the function

$$f(x) = 3x^5 - 5x^3.$$

The derivative, written in its factored form, is

$$f'(x) = 15x^2(x - 1)(x + 1).$$

The solutions of $f'(x) = 0$ are $x = -1$, $x = 0$ and $x = 1$, giving the critical values of the function

$$f(-1) = 2, f(0) = 0 \quad \text{and} \quad f(1) = -2,$$

respectively. The function has the following variations:

$$x < -1, \quad f'(x) = + \text{ and } f(x) \text{ is increasing.}$$
$$-1 < x < \quad 0, \quad f'(x) = - \text{ and } f(x) \text{ is decreasing.}$$
$$0 < x < \quad 1, \quad f'(x) = - \text{ and } f(x) \text{ is decreasing.}$$
$$1 < x \quad\quad , \quad f'(x) = + \text{ and } f(x) \text{ is increasing.}$$

The curve is drawn in Figure 8, showing the critical points at $P_1(-1,2)$, $0(0,0)$ and $P_2(1,-2)$.

18. Velocity.

The speed of a moving body was defined in Section 12 as the rate of change of distance with respect to time. Speed represents a magnitude only, while *velocity* represents both a magnitude and a direction. Hence, the velocity of a moving body may be defined as the *directed speed*.

If the distance of a particle moving on a line from a fixed point of that line is expressed as a function of the time t, the derivative of the function with respect to t is the velocity of the particle at any time. Let s represent the distance and v the velocity of the particle. Then, if

$$s = f(t),$$

$$v = \frac{ds}{dt} = f'(t).$$

A value of t for which v is positive indicates that s is increasing and that the particle is moving to the right. A value of t for which v is negative indicates that s is decreasing and that the particle is moving to the left. And a value of t for which v is zero is the instant at which the particle is at rest.

Suppose that a particle moves on a line so that its distance from a point A at any time is given by the function

$$s = t^3 - 6t^2 + 9t + 4.$$

The velocity of the particle at any time is given by the function

$$v = 3(t^2 - 4t + 3) = 3(t - 1)(t - 3).$$

The particle comes to rest at the end of 1 second and at the end of 3 seconds. We study the direction of motion as follows:

Fig. 9

$t < 1, \quad v = +$ and the particle is moving to the right.

$1 < t < 3, \quad v = -$ and the particle is moving to the left.

$3 < t, \quad v = +$ and the particle is moving to the right.
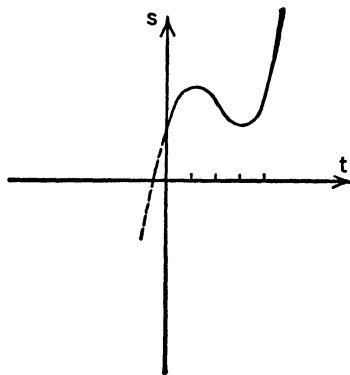
It is instructive to draw the given function with reference to a pair of t- and s-axes as has been done in Figure 9. From such a figure a complete
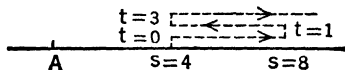
FIG. 10

analysis of the motion may be made and a diagrammatic representation of the motion constructed as in Figure 10.

Exercise 8

GROUP A

1. Find the critical values of the function $f(x) = x^3 - 6x^2 - 15x + 6$.
2. For what values of x is the function $f(x) = x^3 + 3x^2 - 9x + 8$ increasing? Decreasing?
3. Find the coordinates of the points on the curve $y = x^3 + 3x^2 - 24x + 6$ at which the tangents are parallel to the x-axis. For what values of x are the slopes positive?
4. Find the critical values of the function $f(x) = 3x^5 - 5x^2 + 3$. Draw the graph of the function. Show that one root of $f(x) = 0$ is real and that four are imaginary.
5. Find the coordinates of the point on the curve $y = 3x^4 - 8x^3 + 6x^2 + 1$ at which the slope is zero. Test whether the curve is rising or falling to the right and the left of each point determined. Draw a graph of the function using the information obtained.
6. A baseball is thrown upward so that its distance from the ground at any time is given by $s = 128t - 16t^2$. How long and how high does the ball rise? When and with what velocity does it strike the ground?

In each of the following problems s represents the distance of a particle moving on a line from a fixed point of the line at a any time t. Draw a graph of each function and make a complete analysis of the motion.

7. $s = t^2 - 10t + 7$.
8. $s = t^3 - 15t^2 + 48t + 8$.
9. $s = t^4 - 12t^3 + 46t^2 - 60t + 12$.
10. $s = t^2 - 8t + 12$. During what times is the particle approaching the fixed point?

GROUP B

11. If the perimeter of a rectangle is 40 ins., express the area as a function of one side x. For what values of x is the function increasing? Decreasing? Draw a graph of the function and find the dimensions of the rectangle having the greatest area.
12. If the area of a rectangle is 25 sq. ins., express the perimeter as a function of one side x. For what values of x is the function increasing? Decreasing? Draw a graph of the function and find the dimensions of the rectangle having the least perimeter.

13. For what values of x is the derivative of the function $f(x) = x^3 - 4x^2 + 8x + 11$ increasing? Decreasing?

14. Find the critical values of the slope of the curve $y = x^4 + 2x^3 - 12x^2 + 8x - 16$.

15. For what values of x is the slope of the curve $f(x) = x^3 - 6x^2 - 3x + 5$ increasing? Decreasing?

16. Find the coordinates of the points on the curve $f(x) = x^4 - 6x^3 + 12x^2 - 7x + 13$ at which the slope is neither increasing nor decreasing.

17. The path of a punted football is given by the equation $y = 150x - x^2$. If the x-axis represents the ground, how high will the ball rise? How far from the starting point and at what angle does the ball strike the ground?

18. The distance of a particle moving on a line from a fixed point at any time is $s = t^3 - 3t^2 + 8t - 11$. When is the velocity increasing? Decreasing? When neither increasing nor decreasing?

19. Find the angle between the curves $y = x^2$ and $2y = x^2 + 4$ at one of their points of intersection.

20. Prove that the tangents drawn to the parabola $y^2 = 4px$ at the extremities of the chord through the focus perpendicular to the axis are perpendicular to each other.

21. Find the coordinates of the vertex of the parabola $y = x^2 + 4x$ and find its width 14 units above the vertex.

22. Find the coordinates of the vertex of the parabola $y = 2 + 6x - x^2$ and show whether it opens upward or downward.

23. Find the coordinates of the vertex of the parabola $y = ax^2 + bx + c$.

19. Maxima and Minima.

Let $f(x)$ be a continuous single-valued function having a continuous first derivative for a range of values of x including $x = c$. If $f(x)$ is a polynomial, it and its derivative $f'(x)$ are continuous single-valued functions for all values of x. But if $f(x)$ is not a polynomial, it and its derivative are assumed to be continuous within a range $a < x < b$, where c is some value of x within this range. The assumption that the first derivative is continuous within the range means geometrically that the curve has no breaks in it and at no point has a vertical tangent. If $f'(c) = 0$, then $f(c)$ is a critical value of the function.

A maximum and a minimum value of a function $f(x)$ are defined as follows:

The value $f(c)$ is a *maximum value* of the function if there is a range of values of x containing c in its interior such that if x_1 is any other value in that range,

$$f(c) > f(x_1).$$

The value $f(c)$ is a *minimum value* of the function if there is a range of values of x containing c in its interior such that if x_1 is any other value in that range,

$$f(c) < f(x_1).$$

Again, $f(c)$ is a maximum value of the function, if for values of x less than c, but sufficiently near c, $f'(x)$ is positive and for values of x greater than c, but sufficiently near c, $f'(x)$ is negative. Similarly, $f(c)$ is a minimum value of the function, if for values of x less than c, but sufficiently near c, $f'(x)$ is negative and for values of x greater than c, but sufficiently near c, $f'(x)$ is positive. Thus, a function reaches a maximum when it ceases to increase and begins to decrease and, a function reaches a minimum when it ceases to decrease and begins to increase. If neither of these conditions is satisfied, the critical value of the function is *neither a maximum nor a minimum value.*

In studying the critical values of a function it is often helpful to consider the geometrical representations of the function and various values of its derivative.

Consider the curve representing the continuous single-valued function $y = f(x)$. If $f'(c) = 0$, then the point $[c, f(c)]$ is called a *critical point* of the curve. The tangent to the curve at this point is a horizontal line. A critical point is a *maximum point* of the curve if the slope of the curve changes from positive to negative in going from immediately to the left of the critical point to the immediate right of it. Similarly, a critical point is a *minimum point* of the curve if the conditions are exactly reversed, that is, the slope of the curve changes from negative to positive in going through the critical point. If neither of these conditions is satisfied, the critical point is neither a maximum nor a minimum point.

Consider the function

$$f(x) = x + \frac{4}{x}.$$

To find the values of x which give the critical values of the function, the derivative

$$f'(x) = 1 - \frac{4}{x^2}$$

is set equal to zero. The solutions are $x = 2$ and $x = -2$. To test the critical values $f(2)$ and $f(-2)$ for maxima and minima, the derivative is written in its factored form,

$$f'(x) = \frac{(x + 2)(x - 2)}{x^2}.$$

If $\quad x < -2, f'(x) = +$, and if $\quad -2 < x < 0, f'(x) = -$.

Hence, $f(-2)$ is a maximum value of the function.

If $\quad 0 < x < 2, f'(x) = -$, and if $\quad 2 < x, f'(x) = +$.

Hence, $f(2)$ is a minimum value of the function. The curve representing the function is drawn in Figure 11, showing the maximum point at $P_2(-2,-4)$ and the minimum point at $P_1(2,4)$.
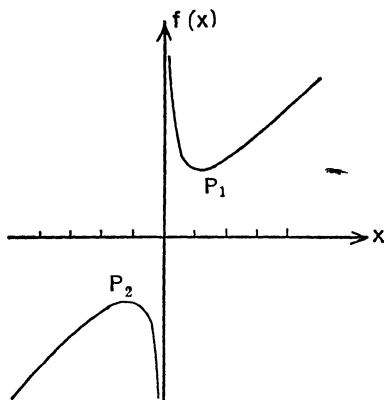
FIG. 11

It may be well to remark that the definition of a maximum value of a function does not state that such a value is greater that any other value of the function, but that it is greater than any other value in its *immediate neighborhood*. A similar statement may be made for a minimum value of a function.

Exercise 9

GROUP A

1. Find the maximum value of the function $f(x) = 2 + 6x - x^2$. Draw the curve and show the test of the maximum value.
2. Find the coordinates of the critical points of the curve $y = x^3 - 9x$. Draw the curve and show tests for maxima and minima.
3. Find the critical values of the function $f(x) = x^4 - 4x^3 - 10$ and give any maximum or minimum value.
4. If $s = t^3 - 9t^2 + 24t + 6$ gives the distance of a particle moving on a line from a fixed point of the line, find the maximum and the minimum distance from the fixed point.
5. Find the minimum velocity of the particle from the equation in Problem 4.
6. Find the minimum slope of the curve $y = x^4 - 2x^3$.
   Draw the graph of each of the following functions.
   7. $f(x) = 4 + 12x + 3x^2 - 2x^3$.
   8. $f(x) = 3x^4 - 8x^3 + 6x^2 + 3$.
   9. $f(x) = 3x^5 - 25x^3 + 60x - 2$.
   10. $f(x) = 5x^6 - 12x^5 - 15x^4 + 40x^3 + 15x^2 - 60x + 5$.

GROUP B

11. Given $f(x) = 2x^3 + 3x^2 - 36x - 85$. By the use of maxima and minima, show that two roots of $f(x) = 0$ are imaginary.

12. Find the critical point of the curve $y = ax^2 + bx + c$. Derive from the result the condition that $ax^2 + bx + c = 0$ have equal roots.

13. Given $f(x) = ax^3 + bx^2 + cx + d$. What kinds of roots may $f'(x) = 0$ have? Show that the equation $f(x) = 0$ has at least one real root.

14. Find two numbers whose sum is 8 and whose product is maximum.

15. Find two numbers whose difference is 12 and whose product is minimum.

16. Find two numbers whose sum is 12 so that the square of one plus twice the other is a minimum.

17. Find two numbers whose sum is 4 so that the cube of one number plus three times the square of the other is a maximum.

18. Find the number which added to its reciprocal gives a minimum sum.

19. Find the number which exceeds its square by a maximum amount.

20. The distance of a particle moving on a line from a fixed point A is $s = 2t^3 - 13t^2 + 20t$. When is the distance from A maximum? When is the distance from A minimum? During what time is the particle approaching the point A?

20. The Second Derivative.

In general, the derivative of a function of a variable is itself a function of that variable and can be differentiated. The derivative of the derivative of a function, each with respect to the independent variable, is called the *second derivative* of the function.

The symbol for the differentiation of a function with respect to x, as used in Section 13, is d/dx. Since the first derivative is represented by either of the symbols dy/dx or $f'(x)$, the second derivative may be represented by either of the following symbols:

$$\frac{d}{dx}\frac{dy}{dx} = \frac{d^2y}{dx^2}, \quad \text{or} \quad \frac{d}{dx}f'(x) = f''(x).$$

The second derivative is the rate of change of the first derivative with respect to the independent variable. Geometrically, the second derivative is the rate of change of the slope of the curve $y = f(x)$ with respect to x.

The third derivative of a function is found by the differentiation of the second derivative. Similarly, higher derivatives are found by continued differentiation. For example, if

$$f(x) = x^4 + 2x^2 + 2,$$
$$f'(x) = 4x^3 + 4x,$$
$$f''(x) = 12x^2 + 4,$$
$$f'''(x) = 24x, \quad f^{\text{iv}}(x) = 24$$

and
$$f^{\text{v}}(x) = 0.$$

Concavity and Inflection Points. The second derivative of a function, being the rate of change of the slope of a curve, gives valuable information regarding the curve itself. Such an investigation is carried out by a study of the sign of the second derivative over the intervals of the independent variable.

If $y = f(x)$, $f'(x)$ *is increasing as x increases for those values of x for which* $f''(x) > 0$. The range of values of x over which the second derivative is positive defines the interval over which the slope of the curve is increasing. As x increases, the tangent to the curve turns in the counterclockwise direction. Hence, in this interval the curve is *concave upward*, as is shown in Figure 12 in the interval $-\infty < x < x_3$. At a minimum point P_1, where $f'(x) = 0$, the second derivative must be positive or zero.
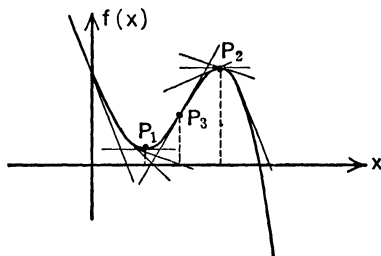
Fig. 12

If $y = f(x)$, $f'(x)$ *is decreasing as x increases for those values of x for which* $f''(x) < 0$. The range of values of x for which the second derivative is negative defines the interval over which the slope of the curve is decreasing. As x increases, the tangent to the curve turns in the clockwise direction. Hence, in this interval the curve is *concave downward*, as is shown in Figure 12 in the interval $x_3 < x < \infty$. At a maximum point P_2, where $f'(x) = 0$, the second derivative must be negative or zero.

If $y = f(x)$, $f'(x)$ *is neither increasing nor decreasing for those values of x for which $f''(x) = 0$*. The points of the curve $y = f(x)$ at which the slope is maximum or minimum locate those points on the curve where the tangent changes from one direction of turning to the other. These are the points at which the curve changes its concavity from downward to upward, or vice versa, and are called *inflection points*. Such a point is P_3 in Figure 12. The abscissas of the inflection points of a curve are real solutions of the equation $f''(x) = 0$, although all such solutions are not necessarily abscissas of inflection points.

Second Derivative Test for Maxima and Minima. In making the test for a maximum or a minimum value of a function it is often more convenient to evaluate the second derivative for the critical value of x, thus determining the sign, than it is to find the variation of the sign in the first derivative. The criteria are as follows:

If at $x = x_1$, $f'(x_1) = 0$ and $f''(x_1) > 0$, the slope is increasing through zero and $[x_1, f(x_1)]$ is a minimum point of the curve.

If at $x = x_2$, $f'(x_2) = 0$ and $f''(x_2) < 0$, the slope is decreasing through zero and $[x_2, f(x_2)]$ is a maximum point of the curve.

If at $x = x_3$, $f'(x_3) = 0$ and $f''(x_3) = 0$, there is no test.

Consider the function

$$f(x) = 3x^5 - 20x^3 + 16.$$

The solutions of the first derivative set equal to zero,

$$f'(x) = 15x^2(x^2 - 4) = 0,$$

are $$x = -2, x = 0 \quad \text{and} \quad x = 2.$$

These values give the critical values of the function. The second derivative is

$$f''(x) = 60x(x^2 - 2),$$

from which $f''(-2) = -240, f''(0) = 0,$ and $f''(2) = 240.$

Hence, $P_1(-2, 80)$ and $P_2(2, -48)$

are maximum and minimum points, respectively. For $x = 0$, the second derivative test fails. But since $f'(x)$ does not change sign as x increases through zero, the point $B(0,16)$ is neither a maximum nor a minimum point.

The solutions of the equation

$$f''(x) = 60x(x^2 - 2) = 0,$$

are

$$x = -\sqrt{2}, x = 0 \quad \text{and} \quad x = \sqrt{2}.$$

Hence, $A(-\sqrt{2}, 16 + 28\sqrt{2})$, $B(0,16)$ and $C(\sqrt{2}, 16 - 28\sqrt{2})$ are the inflection points. The curve is concave downward to the left of A, concave upward between A and B, concave downward between B and C and concave upward to the right of C. The curve is drawn in Figure 13.

Fig. 13

The function $f(x) = x^3 + 2$ has the first and second derivatives

$$f'(x) = 3x^2, \quad f''(x) = 6x.$$

The solution of $f'(x) = 0$ is $x = 0$, giving $f(0) = 2$ a critical value of the function. Since $f''(0) = 0$, the second derivative test fails. As x increases through zero, $f'(x)$ does not change sign and the critical value
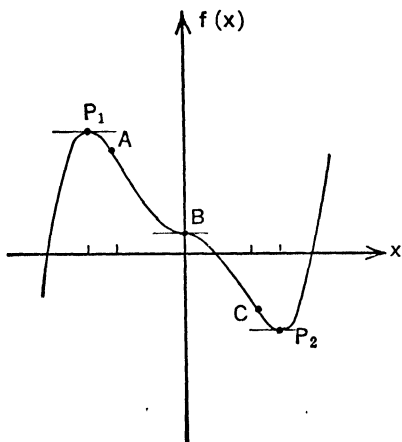
is neither a maximum nor a minimum value of the function. The curve has a horizontal tangent at the point (0,2), is concave downward to the left of it and is concave upward to the right of it.

In comparison with the study of the function just given, let us consider the function $f(x) = x^4 + 2$. It has the first and second derivatives

$$f'(x) = 4x^3, \quad f''(x) = 12x^2.$$

The critical value of the function is $f(0) = 2$ and, here again, the second derivative test fails. However, in this case, as x increases through zero, $f'(x)$ changes from negative values to positive values and the critical value is a minimum value of the function.
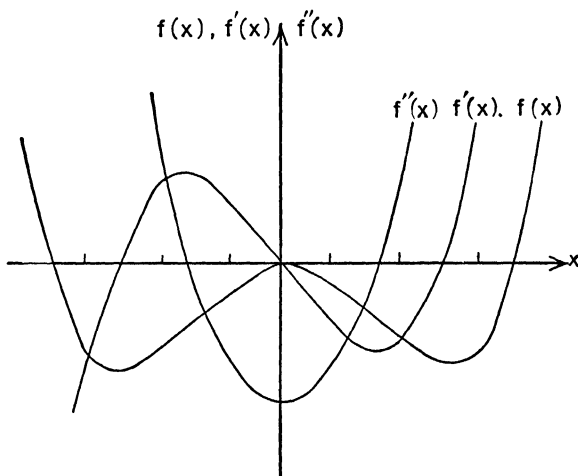
FIG. 14

Derived Curves. It is often instructive to draw in the same figure the graphs of the original function $f(x)$, its derivative $f'(x)$ and its second derivative $f''(x)$. The graphs of the derivatives are called the *derived curves*. When these curves are drawn on the same axes the ordinates represent successively the values of the given function, the slopes of the curve and the rate of change of the slope.

Perhaps the simplest method of procedure in drawing such curves is to construct the second derivative first. From the information obtained from it the first derivative is then constructed. Finally, from the latter curve the original curve may be constructed. In Figure 14 the curve representing the function

$$f(x) = x^4 - 6x^2$$

is drawn from the derived curves obtained from the derivatives,

$$f'(x) = 4x(x^2 - 3),$$
$$f''(x) = 12(x^2 - 1).$$

21. Acceleration.

In Section 18 a study was made of the velocity of a particle moving on a straight line, where the distance of the particle from a fixed point of the line at any time is given as a function of the time.   If

$$s = f(t),$$
$$v = f'(t).$$

The *acceleration* of the moving particle is defined as the rate of change of the velocity with respect to the time.   Letting j represent the acceleration,

$$j = \frac{dv}{dt} = \frac{d^2s}{dt^2} = f''(t).$$

If the acceleration is positive, the particle is moving with an increasing velocity.   If the acceleration is negative, the particle is moving with a decreasing velocity.   But it should be noted that if the velocity is negative, an increasing velocity means a decreasing speed and a decreasing velocity means an increasing speed.

In case the acceleration is constant, a particle is said to be uniformly accelerated.   An important illustration of uniformly accelerated motion is that of a body falling toward the earth from a point near the surface, where air resistance and other such forces are neglected.   The attraction of the earth gives the body an acceleration g which is called the *acceleration of gravity*.   Thus, we write

$$s = \tfrac{1}{2}gt^2, \quad v = gt \quad \text{and} \quad j = g.$$

Exercise 10

GROUP A

1. Find the coordinates of the critical points of the curve $f(x) = 4x^3 + 3x^2 - 18x + 4$ and make the second derivative tests for maxima and minima.

2. Find the coordinates of the inflection point of the curve $f(x) = x^3 - 3x^2 + 6x + 9$ and find its minimum slope?

3. Over what intervals is the curve $f(x) = x^3 - 9x^2 + 8x + 7$ concave upward and downward?

Draw each of the following curves after making the second derivative maxima and minima tests.

4. $y = x^3 - 12x + 4.$

5. $y = 5x^3 - x^2 - 8x + 6.$

6. $y = x^3 + 3x - 3x + 5.$

7. $y = x^4 - 8x + 2.$

In each of the following problems s is the distance of a particle moving on a line from a fixed point at any time t.

8. $s = t^3 - 3t^2 + 2t + 4$. Find the position and the velocity when the acceleration is zero.

9. $s = 2t^3 - 9t^2 + 12t + 5$. For what values of t is the velocity increasing? For what values is the velocity decreasing?

10. $s = t^4 - 6t^3 + 12t^2 + 6t + 4$. Find the maximum and the minimum velocities. Give the maxima and minima tests.

GROUP B

Draw a small arc of the curve $y = f(x)$ through the point P_1 under each of the following conditions.

11. If $f(x_1) = +, f'(x_1) = -$ and $f''(x_1) = +$.
12. If $f(x_1) = +, f'(x_1) = +$ and $f''(x_1) = -$.
13. If $f(x_1) = -, f'(x_1) = -$ and $f''(x_1) = -$.
14. If $f(x_1) = -, f'(x_1) = +$ and $f''(x_1) = +$.
15. If $f(x_1) = +, f'(x_1) = 0$ and $f''(x_1) = +$.
16. If $f(x_1) = -, f'(x_1) = 0$ and $f''(x_1) = -$.

17. In Figure 15 give the signs of the first and the second derivative at each of the points A, B, C, and D.

18. In Figure 16 give the value of the first derivative and the sign of the second derivative at each of the points A, B and C.

19. In Figure 17 give the sign of the first derivative and the value of the second derivative at each of the points A, B and C.

20. Draw on the same axes the curve
$$y = x^3 - 3x$$
and the first and second derived curves.
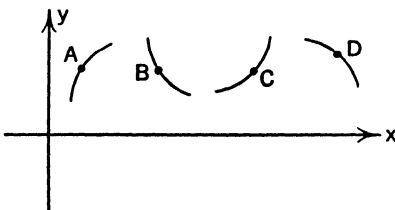
Fig. 15
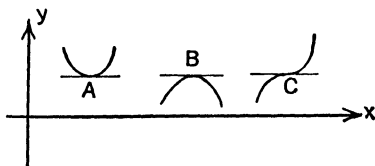
Fig. 16
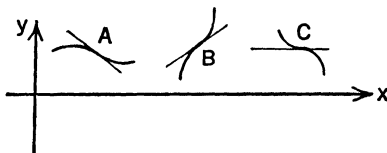
Fig. 17

GROUP C

21. Given $y = x^3 - \frac{3}{5}x^5$ Find coordinates of critical and inflection points. Draw the curve and first derived curve.

Determine the unknown constants in each of the following equations from the data given:

22. $y = ax^3 + bx^2 + cx + d$, if the curve has a minimum point at (1,2) and a maximum point at (0,3).

23. $y = ax^3 + bx^2 + cx + d$, if the curve is tangent to the x-axis at (2,0) and has an inflection point at (0,4).

24. $y = ax^3 + bx^2 + cx + d$, if the curve is concave upward to the left of the origin and concave downward to the right of it and has a critical point at (1,1).

25. $f(x) = ax^3 + bx^2 + cx + d$, if the curve is tangent to $x - y + 4 = 0$ at (0,4) and $f(x) = 0$ has roots -1 and 2.

26. $f(x) = ax^3 + bx^2 + cx + d$, if the curve is tangent to $8x + y - 10 = 0$ at the inflection point $(2,-6)$ and $f(0) = 2$.

27. $y = ax^4 + bx^3 + cx^2 + dx + e$, if the curve has a critical point at (0,2) and a slope of -2 at the inflection point (1,0).

28. Find the equations of the tangents at point (2,2) and at the inflection point of the curve $y = x^3 - 3x^2 + 4x - 2$. Find the acute angle between the tangents.

29. Show that the equation of the tangent to $y = ax^3 + bx^2 + cx + d$ at (x_1,y_1) is $y = (3ax_1^2 + 2bx_1 + c)\,x + (d - bx_1^2 - 2ax_1^3)$.

30. Show that the curve $y = ax^4 + bx^3 + cx^2 + dx + e$ has at least one critical point and either two or no inflection points.

22. Applications of Maxima and Minima.

In Section 19 it was shown that for a real value of x for which the first derivative vanishes, the function assumes a maximum or a minimum value, provided that the first derivative changes sign for this value of x, or, provided that the second derivative does not vanish also for that value of x. In application of this principle, many important problems are concerned with some magnitude which varies continuously, subject to certain restrictions, and require for their solution that the maximum or the minimum value be found. The given conditions enable one to express the measure of the magnitude as a function of a single variable, usually within a restricted interval. The critical values of the function can then be found and examined for maxima and minima.

In some applications it may be found convenient to express the function, whose maximum or minimum value is sought, in terms of two or more variables. Until the method for differentiation of a function of more than one variable is presented, it becomes necessary to eliminate all of the variables except one by means of relations which exist among them as determined from the conditions of the problem.
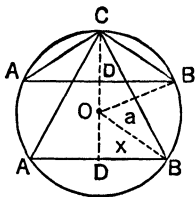
Fig. 18

As a first illustration of the application of the theory of maxima and minima, let us find the dimensions of the right circular cone of maximum volume which can be inscribed in a given sphere.

As in Figure 18, let the altitude DC and the radius of the base DB of the cone be represented by the variables y and x, respectively. Let the

radius of the given sphere be the constant a. Then

$$V = \frac{\pi}{3} x^2 y.$$

From the right triangle OBD, the relation between x and y is

$$x^2 = a^2 - (y - a)^2,$$

or

$$x^2 = a^2 - (a - y)^2.$$

From either

$$x^2 = 2ay - y^2.$$

Eliminating x from the first equation expressing the volume, we have

$$V = \frac{\pi}{3} (2ay^2 - y^3).$$

Since,

$$\frac{dV}{dy} = \frac{\pi}{3} y(4a - 3y),$$

the function V has critical values for $y = 0$ and for $y = 4a/3$. And since,

$$\frac{d^2V}{dy^2} = \frac{\pi}{3} (4a - 6y)$$

if

$$y = \frac{4a}{3}, \frac{d^2V}{dy^2} = - \frac{4a\pi}{3}.$$

Hence, the altitude and the radius of the base of the cone having the maximum volume are

$$\frac{4a}{3} \quad \text{and} \quad \frac{2\sqrt{2}}{3} a,$$

respectively.

As a second illustration, we shall find the most economical proportions for a cylindrical can of given capacity, if no allowance is made for waste of material.

The interpretation of this problem is that the volume of the cylinder shall be constant and that the surface area shall be a minimum in order that the least amount of material be used. Let the variables x and y represent the radius of the base and the altitude, respectively. And let the surface area be S and the volume be a. Then

$$S = 2\pi x y + 2\pi x^2$$

and

$$a = \pi x^2 y.$$

Eliminating y from the first equation by means of the second,

$$S = \frac{2a}{x} + 2\pi x^2.$$

Following the same procedure as before,

$$\frac{dS}{dx} = 4\pi x - \frac{2a}{x^2} = \frac{2}{x^2}(2\pi x^3 - a).$$

$$x = \frac{1}{2}\sqrt[3]{\frac{4a}{\pi}} \quad \text{and} \quad y = \sqrt[3]{\frac{4a}{\pi}}$$

give S minimum. The solution of this problem enables one to state that a cylindrical can whose altitude equals the diameter of its base has the least surface area of all the cylindrical cans of the same volume.

Exercise 11

GROUP A

1. Find the volume of the greatest open box which can be formed from a piece of cardboard 6 ins. square by cutting equal squares from the corners and turning up the edges.

2. A tank with no top is to hold 300π cu. ft. and is to be made in the form of a right circular cylinder, the circular base being horizontal. The material used for the base costs twice as much per sq. ft. as that used for the sides. Find the dimensions of the most economical tank.

3. A poster is to be printed having margins at the top and bottom of the printed matter 4 ins. wide, and margins at each side 3 ins. If the area of the cardboard must be 1728 sq. ins., find its dimensions to give the maximum area of printed matter.

4. A trough is to be made of a long rectangular piece of tin by bending up the two edges so as to give a rectangular cross section. If the width of the piece of tin is 14 ins., find the dimensions of the trough in order that the carrying capacity be a maximum. (The carrying capacity is a maximum when the cross section is a maximum.)

5. Of all the lines which may be drawn through the point (4,1), find the equation of the one for which the sum of the intercepts on the coordinate axes is a minimum. Suggestion: Express the intercepts in terms of the slope.

6. Show that for a given perimeter, the square is the rectangle having the maximum area.

7. Show that for a given area, the square is the rectangle having the minimum perimeter.

8. Find the dimensions of the largest rectangle which can be inscribed in an isosceles triangle, base b and altitude a, one side of the rectangle lying on the base.

9. An open rectangular tank with a square base is to contain a given volume. If no allowance is made for thickness or waste of material, find the dimensions which will require the least amount of material.

10. Find the area of the rectangle having the greatest area which can be inscribed in a right triangle having a given base and altitude, one vertex of the rectangle being at the right angle.

GROUP B

11. Find the dimensions of the most economical cylindrical tin cup which holds a given volume, not allowing for thickness or waste of material.

12. A piece of wire 24 ins. long is to be cut into two pieces, one of which is to bend into the form of a square and the other into the form of an equilateral triangle. Find the lengths of the pieces when the sum of the areas is a minimum.

13. The combined length and girth of a parcel post package is limited to 6 ft. Find the dimensions of the largest rectangular package having a square base which can be posted. Find the dimensions of the largest cylindrical package which can be posted.

14. The strength of a rectangular beam varies as the breadth times the square of the depth. Find the dimensions of the strongest beam which can be cut from a circular log.

15. Find the volume of the maximum right circular cylinder which can be inscribed in a right circular cone, diameter of base b and altitude a.

16. Find the volume of the maximum right circular cylinder which can be inscribed in a sphere of radius a.

17. A metal casting has the form of a cylinder with the ends hollowed out in the form of hemispheres whose radii equal the radius of the cylinder. If the volume of the casting is $5\pi/6$ cu. ins., find the radius and the length of the casting, in order that the cost of finishing the surface be a minimum.

18. A window has the form of a rectangle surmounted by a semicircle having a diameter equal to the width of the window. If the total perimeter of the window is a ft., find the dimensions which will admit the most light.

19. A piece of wire 20 ins. long is to bend so as to enclose a circular sector. Find the radius if the area of the sector is to be as large as possible.

20. A sector is to be cut from a circular piece of tin whose radius is a. From the remaining portion the two edges are to be soldered together forming a right conical funnel. Find the altitude and the radius of the base of the cone if the volume is a maximum

21. A long rectangular strip of tin a ins. wide is to be bent to form an open eaves trough. Assume the two sides to be vertical and the bottom to form a semicircle. In order that the carrying capacity be greatest, show that the tin should be bent so that the cross section is in the form of a semicircle. Find the radius

22. The sum of the two bases and altitude of a trapezoid is a and the difference of the two bases is b. Find the bases and the altitude if the area is a maximum.

23. The equation of the curve of a stream of water projected from a hose is

$$y = mx - \frac{(1 + m^2)x^2}{100},$$

where m is the slope of the nozzle which is taken at the origin. For what value of m will the water reach the greatest height on a wall 45 ft. from the nozzle? Find the greatest height.

24. A coal hopper having a rectangular open top is to be built 50 ft. long. The sides and ends are vertical. The bottom is formed by two planes inclined downward 60° from the horizontal, meeting in a center line. A vertical cross section, which is three sides of a rectangle and two sides of an equilateral triangle, is to have a perimeter of 66 ft. Find the dimensions in order that the capacity of the hopper be greatest.

25. One man starts at a point A and walks 60° N of E at the rate of 3 miles per hour. At the same moment a second man starts at a point B, 20 miles east of A, and walks west toward A at the rate of 4 miles per hour. After how long a time is the square of the distance between them, and hence the distance, a minimum?

CHAPTER III

23. Infinitesimals.

An *infinitesimal is a variable whose limit is zero.* Such a variable may take on any series of values, not all of which are necessarily small. However, *ultimately*, the absolute value of an infinitesimal must become and remain less than any small positive constant, however small. We have illustrations of infinitesimals in the process of differentiation; Δx and Δy are such variables. Contrary to common usage of " infinitesimal " as a constant, however small in value, the mathematical definition requires the use of infinitesimal for *variables only*.

If one infinitesimal is a function of a second, the independent variable is known as the *principal infinitesimal.* Let α and β be two infinitesimals such that

$$\beta = f(\alpha).$$

Then α is the principal infinitesimal.

- **Order of Infinitesimals.** A concept which is of fundamental importance in the study of infinitesimals is the *order* of one with respect to the other. Infinitesimals are separated into classes according to the rapidity with which they approach zero relative to some principal infinitesimal. Suppose that α is the principal infinitesimal which approaches zero by taking on the values indicated in the first line below. The corresponding values of the infinitesimals 2α, α^2 and α^3 are shown in the lines following.

α	:	0.1	0.01	0.001	0.0001
2α	:	0.2	0.02	0.002	0.0002
α^2	:	0.01	0.0001	0.000001	0.00000001
α^3	:	0.001	0.000001	0.000000001	0.000000000001

From this table we can see that there is a great difference between the *relative* behavior of α and 2α as compared with that of α and α^2, and as compared with that of α and α^3. It may be said that 2α is relatively keeping pace with α, while α^2 is decreasing relatively at a much more rapid pace. This illustration indicates that we should put all infinitesimals $k\alpha$ into one

class, $k\alpha^2$ into another class, $k\alpha^3$ into a third class, etc., where k is a constant, not zero.

If
$$\lim_{\alpha \to 0} \frac{\beta}{\alpha} = k, \quad k \neq 0,$$

the infinitesimals α and β are said to be of the same order, or β *is of the first order with respect to* α. On the other hand, if

$$\lim_{\alpha \to 0} \frac{\beta}{\alpha} = 0,$$

β *is an infinitesimal of higher order than* α. If α is an infinitesimal, then $\alpha^2, \alpha^3, \alpha^4 \cdots$, are infinitesimals of higher order than α. Their orders with respect to α are second, third, fourth, $\cdots$, respectively. More generally, if a value of n can be found for which

$$\lim_{\alpha \to 0} \frac{\beta}{\alpha^n} = k,$$

β *is an infinitesimal of the nth order with respect to* α.

To illustrate infinitesimals of the first, second and third order with respect to a principal infinitesimal, let us take the circumference of a circle, the area of a circle and the volume of a sphere, each expressed as a function of the radius.

$$s = 2\pi r, \quad S = \pi r^2, \quad V = \tfrac{4}{3}\pi r^3.$$

Let r approach zero, thus making s, S and V infinitesimals.

Since
$$\lim_{r \to 0} \frac{s}{r} = \lim_{r \to 0} 2\pi = 2\pi,$$

s is of the first order with respect to r.

Since
$$\lim_{r \to 0} \frac{S}{r^2} = \lim_{r \to 0} \pi = \pi,$$

S is of the second order with respect to r.

Finally, since
$$\lim_{r \to 0} \frac{V}{r^3} = \lim_{r \to 0} \tfrac{4}{3}\pi = \tfrac{4}{3}\pi,$$

V is of the third order with respect to r.

Principal Part of an Infinitesimal. An infinitesimal β which is a function of the principal infinitesimal α, frequently is made up of two or more

terms of different orders. The term of the lowest order is called the *principal part of the infinitesimal*. Thus, the order of the infinitesimal β is the same as the order of its principal part.

In general, if

$$\frac{\beta}{\alpha^n} = k + \epsilon, \quad \text{where } \lim_{\alpha \to 0} \frac{\beta}{\alpha^n} = k$$

and where ϵ is an infinitesimal, the term $k\alpha^n$ is the principal part. If $n = 1$, the term $k\alpha$ is the principal part.

Consider a slender cylindrical rod of constant length a and radius of the base r. The total surface area is

$$S = 2\pi ar + 2\pi r^2.$$

As r approaches zero, the area S approaches zero and both variables are infinitesimals. For small values of r, it is obvious that the area of the ends is small as compared with the lateral area. In fact, the lateral area is an infinitesimal of the first order and the area of the ends is an infinitesimal of the second order, each with respect to the radius. The infinitesimal S is of the same order as r and the lateral area, $2\pi ar$, is its principal part.

If V is the volume of a cube whose edge is x, then $V = x^3$. Let the side be increased by a length Δx. Then the volume V is increased by

$$\Delta V = 3x^2 \, \Delta x + \Delta x \, (3x \, \Delta x + \overline{\Delta x}^2).$$

If we choose a value of x, and let Δx approach zero, the variable ΔV is an infinitesimal depending on the principal infinitesimal Δx. And since,

$$\frac{\Delta V}{\Delta x} = 3x^2 + (3x \, \Delta x + \overline{\Delta x}^2)$$

and

$$\lim_{\Delta x \to 0} \frac{\Delta V}{\Delta x} = 3x^2,$$

Δx and ΔV are infinitesimals of the same order. Moreover the term $3x^2 \, \Delta x$ is the principal part of ΔV.

Exercise 12

GROUP A

1. A straight steel rod is a ins long and has a square cross section whose side is x. Express the perimeter p of one end, the total surface area S and the volume V of the rod as functions of x. Compute each of the values of p, S and V for $x = 1$, $x = 0.1$ and $x = 0.01$ ins.

2. In Problem 1, find the orders of the infinitesimals p and V as x approaches zero.

3. In Problem 1, find the order of S and the orders of its two parts as x approaches zero. Find the order of each of the following infinitesimals.

    4. The surface area of a sphere with respect to its radius.

    5. The convex surface area of a right circular cylinder of constant height with respect to the radius of the base

    6. The volume of a right circular cone having altitude a, with respect to the radius of its base.

    7. The surface area of a cube with respect to its edge.

8. Find the order and the principal part of the total surface area of a right circular cone whose altitude equals the radius of the base as the radius of the base approaches zero.

9. Find the order and the principal part of the function $f(x) = 2x^2 - 6x$ as x approaches zero.

10. Given the function $y = x^2 + 6x + 1$. Find Δy for any Δx and find the order and principal part of Δy with respect to Δx as Δx approaches zero.

GROUP B

Find the order and the principal part of each of the following infinitesimals.

    11. The volume of a rectangular solid having a square base whose altitude is a units longer than the side of the base, as the side of the base approaches zero.

    12. The surface area of the solid in Problem 11, with respect to the side of the base.

    13. The total surface area of a right prism of length a, whose base is an equilateral triangle, with respect to the side of the base as this side approaches zero.

    14. The volume of the frustum of a right circular cone whose larger base is constant and whose altitude is equal to the diameter of the smaller base, with respect to the radius of the smaller base, as the radius of this base approaches zero.

Find Δy for any Δx and find the order and the principal part of Δy with respect to Δx, as Δx approaches zero for each of the following functions.

    15. $y = 3 - 2x - x^2$.   **17.** $y = x^4 + 2x$.

    16. $y = x^3 - x^2 + 8$.   **18.** $y = x^3 - 2 - x^4$.

If α and β are infinitesimals of the same order and $\lim \alpha/\beta \neq 1$, find the order of each of the following with respect to either α or β.

    19. $\alpha + \beta$.   **21.** $\alpha^2 + \beta^2$.

    20. $\alpha - \beta$.   **22.** $\alpha^2 - \beta^2$.

If α and β are infinitesimals and if β is of the second order with respect to α, find the order of each of the following with respect to α.

    23. $\alpha + \beta$.   **25.** $\alpha \cdot \beta$.

    24. $\dfrac{\beta}{\alpha}$.   **26.** $\dfrac{\alpha^3}{\beta}$.

24. Differentials.

As has been said, the variables Δx and Δy as used in the fundamental differentiation process, are infinitesimals. The derivative of a continuous single-valued function $y = f(x)$ has been defined as the limit of the difference-quotient,

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = f'(x).$$

In general, $f'(x) \neq 0$. This expresses the fact that Δx *and* Δy *are infinitesimals of the same order.*

In general, the infinitesimal Δy is composed of infinitesimals of the first order and higher. Thus, we may write

$$\frac{\Delta y}{\Delta x} = f'(x) + \epsilon,$$

or $\qquad \Delta y = f'(x)\,\Delta x + \epsilon\,\Delta x.$

In this latter expression, Δx is the principal infinitesimal and $f'(x)$ does not vary with respect to it, that is, $f'(x)$ is fixed for a fixed x. Therefore, the *term* $f'(x)\,\Delta x$ *is the principal part of* Δy. The term $\epsilon\,\Delta x$ represents all infinitesimals of higher order.

The *principal part of the increment of a function* y *is represented by the symbol* **dy** *and is called the* **differential** *of the function.* Thus, if $f'(x) \neq 0$, the differential of $f(x)$ is defined by

$$dy = f'(x)\,\Delta x.$$

Hereafter, for the sake of symmetry, when expressing the differential of the dependent variable dy, we shall define the *differential dx* of the independent variable as the increment Δx. Thus, we write

$$\boldsymbol{dy = f'(x)\,dx.}$$

For the function $y = x^3 + 2x^2$, $\dfrac{dy}{dx} = 3x^2 + 4x$. Hence, by the definition of the differential of the function,

$$dy = (3x^2 + 4x)\,dx.$$

The increment of the given function, found in the usual way by giving x an increment, is

$$\Delta y = (3x^2 + 4x)\,\Delta x + (3x\,\Delta x + 2\,\Delta x + \overline{\Delta x}^2)\,\Delta x.$$

From this it can be seen that the principal part of Δy is dy.

Let us further illustrate the difference between Δy and dy by comparing their numerical values for small values of the increment of the independent variable, using the function given above. Let $x = 10$, giving

$$dy = 340\,dx \quad \text{and} \quad \Delta y = 340\,\Delta x + (32\,\Delta x + \overline{\Delta x}^2)\,\Delta x.$$

If $\Delta x = dx = 0.1$, $dy = 34$ and $\Delta y = 34 + 0.321$

If $\Delta x = dx = 0.01$, $dy = 3.4$ and $\Delta y = 3.4 + 0.003201$

If $\Delta x = dx = 0.001$, $dy = 0.34$ and $\Delta y = 0.34 + 0.000032001$

Having defined the differential of a function $y = f(x)$ as $dy = f'(x)\,dx$, we may now take the quotient of the differentials (dy) and (dx) and obtain

$$\frac{(dy)}{(dx)} = f'(x) = \left(\frac{dy}{dx}\right).$$

This justifies our original choice of the latter symbol as one of the representations of the derivative.

Differentials may be used in the technique of differentiation. The advantage of their use, however, is not at once apparent but is indicated in a later chapter. The differentiation of the explicit function $y = x^3 - 3x^2$ by differentials gives

$$dy = 3x^2\,dx - 6x\,dx$$

and by the derivative gives $dy/dx = 3x^2 - 6x$.

25. Geometric Interpretation of Differentials.

In Figure 19 let $y = f(x)$ be represented by the curve PP'. At any point $P(x,y)$ on the curve, x is given an increment

$$\Delta x = PQ \quad \text{then} \quad \Delta y = QP'$$

Fig. 19

is the corresponding increment of the function. The tangent PR is drawn to the curve at the point P, having an inclination θ.

Then $\tan \theta = f'(x)$.

From the right triangle PQR,

$$QR = PQ \tan \theta.$$

or $QR = f'(x)\,\Delta x.$

Hence, $QR = dy.$
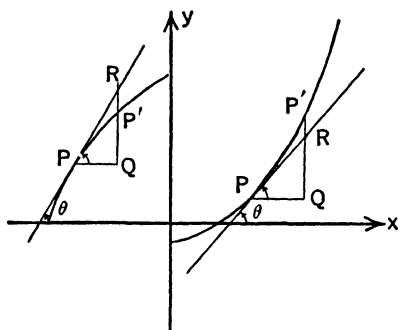
Exercise 13

GROUP A

1. Given $y = 2x - x^2$. Find Δy, the principal part of Δy and $\Delta y - dy$.
2. Given $y = x^3 + x$. Show that the principal part of Δy is the same order as Δx and show that the remainder of Δy is an infinitesimal of higher order than Δx.
3. Given $y = x^3 - x^2$. Find Δy, dy and $\Delta y - dy$.
4. If V and S are the volume and the surface area of a cube whose edge is x, find $\Delta V - dV$ and $\Delta S - dS$.
5. If S is the area of a circle of radius r, find ΔS and dS.
6. In Problem 5 show that dS is equal to the area of a rectangle whose base is the circumference of the circle and whose altitude is dr, and show that ΔS is the difference in the areas of the circles having radii $r + dr$ and r.
7. Given $y = x^2 - 4x$. Find Δy and dy for $x = 1$ and $dx = 0.01$.
8. Given $y = 3x - x^3$. Find $\Delta y - dy$ for $x = 1$ and $dx = 0.002$.
9. If $f(x) = 3 - 6x^2 + 2x^4$, find the differential of the function.
10. If $f(t) = 3t^3 - 2t^2 + 6t + 3$, find the differential of the function.

GROUP B

A particle moves on a line so that its distance s from a fixed point of the line is a function of the time t, its velocity is v and its acceleration is j.

11. Show that $ds = v\,dt$.
12. Show that $dv = j\,dt$.

Find the differential of each of the following functions.

13. $f(x) = \sqrt{x} - x^3$

14. $f(x) = \sqrt{x} + x + \dfrac{1}{\sqrt{x}}$.

15. $f(x) = \dfrac{1}{x^2} - \dfrac{1}{x} + \dfrac{1}{\sqrt{x^3}}$.

16. $f(t) = 2t + \dfrac{1}{t}$.

17. $v = t^2 - 3t^3$.

18. $s = t^5 - \sqrt{t}$.

19. If $y = 2/x$, show that Δy is made up of two infinitesimals, one of the first order and one of the second order with respect to Δx.
20. If $y = x - (1/x)$, compute the difference between Δy and dy for $x = 2$ and $dx = 0.001$.
21. If the radius of a circle is 6 ins. and if it is increased by 0.02 in., compute the difference between the areas ΔS and dS.
22. If the radius of a sphere is 10 ins. and if it is increased by 0.03 in., compute the difference between the volumes ΔV and dV.

26. Parametric Equations.

In analytic geometry it is frequently found to be more convenient to express the locus of a point by means of two equations, one for x and another for y in terms of a third variable, rather than the single Cartesian equation in terms of x and y. The third variable is known as the *parameter*, and

the equations are called the *parametric equations* of the locus. The ordinary Cartesian equation of a curve can often be found from the parametric equations by the elimination of the parameter. However, there are many cases in which such eliminations are either complicated or even impossible to obtain. In fact, parametric equations are most useful in those cases in which it is difficult or impossible to express the Cartesian equation.

If $$y = f_1(t) \quad \text{and} \quad x = f_2(t),$$

we have two functions in which t is the independent variable.

Hence, $$dy = f_1'(t)\, dt \quad \text{and} \quad dx = f_2'(t)\, dt.$$

When dealing with parametric equations, it is necessary to be able to find the derivative of y with respect to x, without having to find the Cartesian equation. We proceed to show that

$$\frac{dy}{dx} = dy \div dx,$$

or that $$\frac{dy}{dx} = \frac{f_1'(t)}{f_2'(t)}, \quad \text{provided that } f_2'(t) \neq 0.$$

From the two given functions we have

$$\Delta y = f_1'(t)\, \Delta t + \epsilon_1\, \Delta t$$

and $$\Delta x = f_2'(t)\, \Delta t + \epsilon_2\, \Delta t,$$

where $\epsilon_1\, \Delta t$ and $\epsilon_2\, \Delta t$ represent infinitesimals of higher order than Δt. Then

$$\frac{\Delta y}{\Delta x} = \frac{f_1'(t) + \epsilon_1}{f_2'(t) + \epsilon_2},$$

and $$\frac{dy}{dx} = \frac{\lim_{\Delta t \to 0} [f_1'(t) + \epsilon_1]}{\lim_{\Delta t \to 0} [f_2'(t) + \epsilon_2]} = \frac{f_1'(t)}{f_2'(t)}, \quad \text{where} \quad f_2'(t) \neq 0.$$

In illustration, consider the equations

$$y = t^2 + 2t, \quad x = 4t^3.$$

Then $$dy = (2t + 2)\, dt \quad \text{and} \quad dx = 12t^2\, dt,$$

and $$\frac{dy}{dx} = \frac{t + 1}{6t^2}.$$

It may be advantageous to show this result in detail as was done in the

general case as follows:

$$\Delta y = (2t + 2) \Delta t + \overline{\Delta t}^2$$

$$\Delta x = 12t^2 \Delta t + (12t \Delta t + \overline{\Delta t}^2) \Delta t.$$

And since $\quad \lim\limits_{\Delta t \to 0} \dfrac{\Delta y}{\Delta x} = \dfrac{\lim [(2t + 2) + \Delta t]}{\lim [12t^2 + (12t \Delta t + \overline{\Delta t}^2)]},$

the same result is obtained as before.

Exercise 14

Find the derivative of y with respect to x for each of the following parametric equations.

1. $y = t^3$, $x = 3t^2$.
2. $y = 5t^2$, $x = t^5$.
3. $y = 2t^3$, $x = t^4$.
4. $y = 2z + z^2$, $x = 6z$.
5. $y = 3r^2 - r$, $x = 1/r$.

Find the equation of the tangent to each of the following curves at the specified points.

6. $y = 2t$, $x = t^2 - t$, where $t = 2$.
7. $y = 3t^2$, $x = t^3 + t$, where $t = 1$.
8. $y = \dfrac{4}{t} - t^2$, $x = \sqrt{t}$, where $t = 4$.
9. $y = \sqrt{z} + z^3$, $x = 1/z$, where $z = 1$.
10. $y = r - 1$, $x^2 = r^3$, where $r = 4$.

27. Approximation by Differentials.

It has been shown that the differential and the increment of a function differ by a very small amount for *small* values of the increment of the independent variable. As a result, the differential of a function may often be used to calculate approximations of changes of values of functions which are produced by small changes in the independent variable.

To illustrate the use of differentials in making approximations, we shall compute the actual area of a square whose side is 3.002 inches and then approximate the area by the use of differentials, noting the difference in the two results obtained.

If x is the length of the side and S is the area,

$$S = x^2.$$

For $\qquad\qquad$ $\Delta S = 2x\,\Delta x + \overline{\Delta x}^2,$

we shall assume $\qquad$ $x = 3 \quad$ and $\quad \Delta x = 0.002$ ins.

Therefore, $\qquad\qquad$ $\Delta S = 0.012004$ sq. ins.

Hence, the exact area of the square is

$$S + \Delta S = 9.012004 \text{ sq. ins.}$$

However, if we use differentials,

$$dS = 2x\,dx = 0.012 \text{ sq. ins.}$$

And hence, the approximate area is

$$S + dS = 9.012 \text{ sq. ins.}$$

We observe that the two results differ by the small amount

$$0.000004 \text{ sq. in.}$$

As a further illustration, let us approximate the value of

$$f(x) = x^3 - 2x^2 + 5x + 1$$

if $x = 1.998$.

If $\qquad\qquad\qquad$ $x = 2, \quad f(2) = 11.$

But since x differs from 2 by a small amount, we may compute the approximate difference of the function which is due to this small change in x. The change in x is

$$dx = -0.002.$$

Since $\qquad\qquad$ $dy = (3x^2 - 4x + 5)\,dx,$

$$dy = f'(2)\,dx = 9(-0.002) = -0.018.$$

Therefore, the approximate value of the function is

$$f(2) + dy = 11 - 0.018 = 10.982.$$

Small Errors. A second application of differentials is made when small errors in calculation are to be computed. For example, suppose that the radius of a circle is measured and found to be 10.05 inches with a maximum error of 0.02 in. We wish to *approximate* the *maximum error* introduced in the computation of the area of the circle. Using S and r for the area

and radius of the circle, respectively,

$$S = \pi r^2.$$

The exact maximum error in S will be the change ΔS found when r changes from 10.05 to 10.07. But the approximate maximum error in S will be the corresponding value of dS, where $dr = 0.02$. Hence, we have

$$dS = 2\pi r \, dr$$

$$dS = 2\pi(10.05)(0.02) = 1.26 \text{ sq. in.}$$

Relative and Percentage Error. In any approximate computation, the error of the result is the amount by which the computed value differs from the true value. This error should always be well within the limit of error allowable for the problem at hand.

If Δy is the error in y, $\Delta y/y$ is called the *relative error* in y, and $100 \, \Delta y/y$ is the *percentage error* in y. These errors may be approximated by the use of differentials instead of increments. Hence,

$$\frac{dy}{y} = \textbf{Relative error in } y\textbf{, approximately,}$$

and

$$100 \, \frac{dy}{y} = \textbf{Percentage error in } y\textbf{, approximately.}$$

Consider the problem last discussed, in which we now wish to approximate the relative error in the computed area of the circle due to the maximum error of 0.02 in. in the measurement of the radius.

$$\frac{dS}{S} = \frac{2(10.05)(0.02)\pi}{(10.05)^2\pi} = 0.00398.$$

Hence, the area is subject to relative maximum error of approximately

0.4 per cent.

Exercise 15

GROUP A

1. Compute both ΔS and dS for the area S of a circle of radius 10 ins. corresponding to an increase of 0.02 in the radius.
2. Find the approximate area of a square whose side is 5.001 ins.
3. Compute the approximate volume of a cube whose edge is 3.002 ins.
4. Compute the approximate volume of a sphere whose radius is 4.001 ins.

5. Find the approximate maximum relative error and the percentage maximum error in the area of a square if its side is measured as 3 ins. with a possible maximum error of 0.01 in.
6. Given $y = x^3 - 3x^2 + 3x + 2$. Find the approximate value of y for $x = 4.99$.
7. Given $f(x) = x^3 - 2x^2 + 2x - 1$. Find approximately $f(3.002)$.
8. Find approximately the volume of a thin spherical shell.
9. Find approximately the square root of 100.8.
10. Find approximately the cube root of 999.1.

GROUP B

11. The distance of a particle moving on a line from a fixed point of that line at any time is given by $s = t^4 - 2t^2 + 8$. Compute, approximately, the distance, the velocity and the acceleration when $t = 2.001$.
12. The altitude of a right circular cylinder is equal to the diameter of its base. If the radius of the base is measured as 5 ins. with a maximum error of 0.005 in., find the approximate maximum errors in the volume and the area of the convex surface.
13. The altitude of a right circular cone is equal to the diameter of its base. If the radius of the base is 10.005 ins., compute the approximate volume and total surface area. Find the approximate relative error in the volume.
14. The altitude of a right circular cone is equal to the radius of its base If the radius is measured to be 6 ins. with a possible maximum error of 0.002 in., find approximate maximum error in the computed area of the convex surface. Compute the approximate percentage maximum relative error in the convex surface
15. Find approximately the change in the reciprocal of a number corresponding to a small change in the number.
16. Spherical iron castings are to be burnished to smooth surfaces. If the process may change the radius by the amount 0.001 in and the allowable change in volume is 1 cu. in., find the radius of the largest sphere for which the process may be used.
17. A closed rectangular box $6 \times 8 \times 10$ ins. is to be lined with lead as a container for radio active compounds. If the lining is $\frac{1}{16}$ in thick, find the approximate change of volume.
18. Spherical shot weighing 10 lbs. each, are made of iron weighing 400 lbs. per cu. ft. If the weight must be accurate to 0.1 lb., find the radius of the shot.
19. If $w = \sqrt{x} + \sqrt[3]{y} + \sqrt[3]{z}$, compute the approximate value of w for $x = 398$, $y = 67$ and $z = 727$.
20. The pressure of a gas varies inversely as the volume of the container. If the allowable error in the pressure is $0.002\ k$, where k is the constant of variation, and the volume can be measured to 0.2 cu. ft., find the size of the smallest container to which the process can be applied.
21. A wooden sphere of radius r is to be cut from a cubical block having an edge of $2r$. If the value of r has a maximum inaccuracy of 1 per cent, find the approximate maximum relative error in the volume of wood remaining.
22. The strength of a rectangular beam varies directly as the product of its breadth and the square of its depth. From a log of radius 6 ins. a beam having a breadth of 3.01 ins. is to be cut. Find the approximate strength of the beam in terms of k, the constant of variation.

GROUP C

23. At any time t, the distance s of a particle moving on a straight line from a point A is given by $s = t^3 - 3t^2 - 144t + 432$. Find the intervals of time when the particle is moving toward A, when the velocity is increasing and when the speed is decreasing.

24. Find the area of the triangle formed by the x-axis and the tangents to the curve $y = 4 - x^2$ at the points for which $x = 2$ and $x = -2$.

25. A coal bin with a square base is to be built in a corner of a basement using two of its walls as sides of the bin. The floor of the bin will cost 20¢ per sq. ft. to build and the two sides to be built will cost 15¢ per sq. ft. Find the cost of the most economical dimensions for a bin whose volume is to be 288 cu. ft.

CHAPTER IV

28. Integration.

The integral calculus is primarily concerned with the problem of finding a function when the derivative of that function with respect to its independent variable is given. Such a function can often be found by inspection and the result verified by differentiation.

Suppose that we are given

$$\frac{dy}{dx} = 6x^2 + 2.$$

Then by inspection, the function which gives this derivative is

$$y = 2x^3 + 2x + C.$$

If C is any constant, its derivative is zero. Hence, the derivative of the latter function is the given function, regardless of the value of C.

An indefinite integral of a function is a function whose derivative with respect to the independent variable is the given function.

In general, if

$$\frac{dy}{dx} = f(x),$$

$$y = F(x) + C,$$

where

$$\frac{d}{dx}[F(x)] = f(x).$$

The process of finding an integral of a function is called *integration*. Consequently, integration and differentiation are inverse processes.

It is customary to express the relationship between the functions $f(x)$ and $F(x)$ by writing

$$\int f(x) \, dx = F(x) + C,$$

where the symbol $\int$ is called the *integration sign*. It is to be observed that

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the integration sign is always followed by the *differential* of the required function and not by its derivative. Thus, if we have

$$y = x^3 + 2, \quad dy = 3x^2 \, dx.$$

Then
$$y = \int dy = \int 3x^2 \, dx = x^3 + C.$$

Although further reasons appear in a later chapter for this demand, here,

$$\int f(x) \, dx$$

requires that we find a function whose derivative with respect to x is $f(x)$. If, however, we have

$$\int y \, dx,$$

it is implied that y is a function of x which must be substituted before the integration can be performed. If

$$y = x^2 - 2x^3 + 1,$$

$$\int y \, dx = \int (x^2 - 2x^3 + 1) \, dx,$$

$$= \frac{x^3}{3} - \frac{x^4}{2} + x + C.$$

The constant C which appears in the integration of a function is known as the *constant of integration*. The value of this constant is independent of the integration process. Consequently, there are as many integrals of a given function as there are values which may be assigned to the constant C.

Thus,
$$y = F(x) + C$$

is a system of functions having the same derivative or the same differential. It is for this reason that such an integral is called an *indefinite integral*. In the applications the value of C is determined from the data of the problem at hand.

29. Integration Formulas.

The differentiation of a function is a direct process. On the other hand, the integration of a function, being the inverse of differentiation, is more or less a matter of trial. That is, the function is found if its derivative is the given function.

From the definition of an integral of a function, the following three formulas are stated:

Integral of x to a Power.   Let n be any constant *except* -1.   Then

(1)
$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C.$$

Integral of a Constant times a Function.   Let a be any constant and let u be any function of x which can be integrated.   Then

(2)
$$\int au \, dx = a \int u \, dx.$$

This formula shows that any constant may be brought outside the integral sign, and conversely.

Integral of a Sum.   Let u and v be any two functions of x which can be integrated.   Then

(3)
$$\int (u + v) \, dx = \int u \, dx + \int v \, dx.$$

Formulated in words, this formula shows that the integral of an algebraic sum of two or more functions is equal to the same algebraic sum of the integrals of those functions.

Exercise 16

GROUP A

Evaluate each of the following integrals and verify each result.

1. $\int (3x^2 - 6x + 2) \, dx.$ **4.** $\int (x + 2)^2 \, dx.$

2. $\int (1 - 3x + x^2 - x^3) \, dx.$ **5.** $\int 5(x + 3)^2 \, dx.$

3. $\int (a + bx) \, dx.$ **6.** $\int 2x(x^2 - 2)^2 \, dx.$

7. If the derivative of $f(x)$ with respect to x is $2x^2 - x + 1$, find $f(x)$.

8. Find $\int dy$ if $dy = (x + 1)^3 \, dx.$

9. If the rate of change of a function with respect to x is $9x^3 + 5x^2 - 8x + 3$, find the function.

10. If the velocity of a particle moving on a line at any time is given by $v = t^2 - 4t + 6$, find an expression for the distance of the particle from a fixed point of the line at any time t.

GROUP B

Evaluate each of the following integrals and verify each result.

11. $\int (3t^3 - 2t^2 + 6t + 8)\, dt.$

12. $\int (2y + 1)\, dx,$ where $y = x^2 - x + 2.$

13. $\int (x - 1 + \sqrt{x})\, dx.$

14. $\int (x^{-2} + 6x + 1 - x^4)\, dx.$

15. $\int 5x^2(2x - x^2)\, dx.$

16. $\int \sqrt{2z^2}(t - 1)\, dt,$ where $z = 3t - 4.$

17. The slope of a system of curves at any point is $2x - 1$. Find the equation of the system.

18. A curve passes through the point $(1,6)$ and has a slope of $3x^2 - 2x$ at any point. Find the equation of the curve.

19. A particle moving on a line is 2 units from a fixed point on that line when $t = 0$. If the velocity at any time t is given by $v = 2t^2 - 3t + 6$, find the distance from the fixed point at any time t.

20. A particle moves on a line so that its velocity at any time t is given by $v = 3t^2 + 4t + 2$. How far will the particle move from $t = 1$ to $t = 3$?

30. Determination of the Constant of Integration.

When a problem of geometry or physics can be formulated by express-ing a function as the rate of change of a variable, the solution can often be found by the integration of that function. It was shown in the last sec-tion that if the derivative or the differential of a polynomial is known, in general, the function itself is known, except for the constant of integration. If, however, a value of the function is known for a particular value of the variable, the constant of integration can be evaluated and thus completely determine the function.

Geometric Applications. If it is known that the rate of change of y with respect to x is $f(x)$,

$$\frac{dy}{dx} = f(x) \quad \text{and} \quad dy = f(x)\, dx.$$

Hence, the integral

$$y = \int f(x)\, dx = F(x) + C,$$

is not a single function, but is a *system of functions* all of which have the same derivative. Consequently, if the derivative is interpreted as the slope of a curve at any point whose abscissa is x, the integral is represented geometrically, not by a single curve, but by a *system of curves*.

If the slope of a curve at any point (x,y) is given to be $2x + 2$,

$$\frac{dy}{dx} = 2x + 2 \quad \text{and} \quad dy = (2x + 2)\, dx.$$

Then
$$y = \int (2x + 2)\, dx$$

and
$$y = x^2 + 2x + C.$$

This equation is represented by a system of parabolas having the line $x + 1 = 0$ as axes. Three curves of the system are drawn in Figure 20.
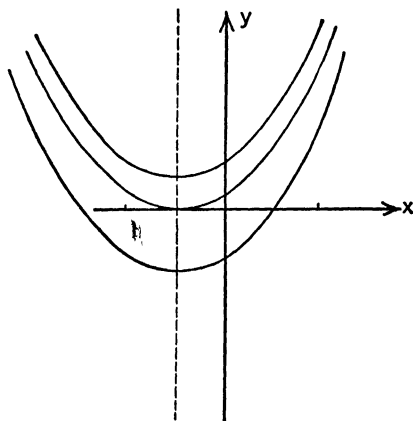
Fig. 20

In the given illustration, suppose that the curve be required to pass through the point (2,9). This restriction demands that C take on a particular numerical value in order that the coordinates of the point satisfy the equation.

Thus, $9 = 4 + 4 + C$, $C = 1$,

giving the unique solution of the problem

$$y = x^2 + 2x + 1.$$

In Figure 20, this equation is represented by the parabola which is tangent to the x-axis.

Physical Applications. If a particle moves in a straight line so that its distance s from a fixed point is a function of the time t, the derivative of s with respect to t is the velocity v and the derivative of v with respect to t is the acceleration j. In this section it is desired to consider some problems involving relations among these magnitudes which are the reverse of those previously encountered.

Suppose that it is known that the velocity of a particle moving on a line in terms of seconds elapsed is

$$v = 2t - 1$$

and that it is desired to find the distance passed over from $t = 2$ to $t = 4$ seconds.

Since
$$ds = v\, dt,$$
$$s = \int (2t - 1)\, dt$$

and
$$s = t^2 - t + C.$$

Assume
$$s = 0, \quad \text{when} \quad t = 2 \text{ secs.}$$

Then
$$0 = 4 - 2 + C \quad \text{and} \quad C = -2$$

making
$$s = t^2 - t - 2.$$

This is the distance from the fixed point at any time t. Accordingly, if $t = 4$,

$$s = 16 - 4 - 2 = 10 \text{ linear units.}$$

In the determination of the value of C above, it is to be noted that its value depends upon the position of the point from which s is measured. This point is not located in the problem. Consequently it may be chosen at pleasure. In assuming $s = 0$ when $t = 2$, the fixed point was chosen as the position of the particle at the instant when 2 seconds had passed.

If a particle moves on a line, it may be that it moves to the left and to the right as illustrated in Section 18. The distance from the fixed point may be increasing at certain times and decreasing at others and distance may become negative. Under such circumstances, distances moved in different directions must be computed separately.

Certain problems require two integrations for their solution. If so, it must be remembered that a constant of integration is introduced each time an integral is taken.

It is given that a particle moves on a line so that its acceleration at any time t is $6t$ feet per second per second and that its distances from the fixed point are 10 and 15 ft. at the end of 2 and 3 seconds, respectively. It is required to find the velocity and the distance from the fixed point at the end of 5 seconds.

Since
$$j = \frac{dv}{dt} \quad \text{and} \quad dv = j\, dt,$$
$$v = \int 6t\, dt = 3t^2 + C_1.$$

Then
$$ds = (3t^2 + C_1)\, dt$$
$$s = \int (3t^2 + C_1)\, dt = t^3 + C_1 t + C_2.$$

If $t = 2$, $10 = 8 + 2C_1 + C_2$.

If $t = 3$, $15 = 27 + 3C_1 + C_2$.

Solving these equations simultaneously, $C_1 = -14$ and $C_2 = 30$. Replacing these constants by their values,

$$v = 3t^2 - 14$$

and $$s = t^3 - 14t + 30.$$

If $t = 5$, $v = 61$ ft./sec. and $s = 85$ ft.

Exercise 17

GROUP A

1. Find the equation of the system of curves which have slopes at any point (x,y) equal to $3x^2 - 2x - 1$. Show that the maximum points of these curves lie on a line parallel to the y-axis and that the minimum points lie on a line also parallel to the y-axis.
2. Find the equation of the curve through the point $(-3,2)$ whose slope at any point (x,y) is $3x^2 - 2x + 4$.
3. It being given that the acceleration due to gravity is 32 ft. per sec. per sec., find the distance from the starting point of a freely falling body in terms of the seconds of falling.

In the following problems v is the velocity of a particle moving on a line at any time t in seconds and j is its acceleration.

4. If $v = 2t + 2$, find the distance covered from $t = 2$ to $t = 4$ secs.
5. If $v = t^2 + 2t + 1$, find the distance covered during the first three seconds.
6. If $v = t^2 - 4t + 3$, find the distances covered during the first, second and the third seconds
7. If $j = 2t - 3$ and if $v = 2$ when $t = 0$, find the distance covered during the second second.
8. If $j = 2t + 4$ and if $v = 2$ when $t = 0$ and $s = 6$ when $t = 3$, find the velocity and the distance after 5 secs.

9. Find the equation of the curve through the point $(4,6)$ whose slope at any point (x,y) is $2\sqrt{x}$.
10. Find the equation of the curve through the points $(0,6)$ and $(3,2)$ if the rate of change of the slope at any point (x,y) is $2x - 3$.

GROUP B

11. The slope of a curve at any point (x,y) is $27 - 3x^2$ and the curve has an x-intercept equal to 1. Draw the curve.
12. Find the coordinates of the maximum and minimum points and the inflection point of the curve whose slope at any point (x,y) is $x^2 - 6x + 8$ and whose y-intercept is -2.

13. Find the equation of the system of curves having the slope at any point (x,y) equal to 1 divided by the square root of the abscissa of the point. Draw three curves of the system. Find the equation of the curve of the system which is tangent to the y-axis.

14. A body is thrown downward with an initial velocity of 20 ft./sec. Assuming the acceleration as 32 ft /sec.², find the velocities after 2 secs. and after 3 secs. Find the distance covered during the first two seconds.

15. A body is thrown upward from an elevation with an initial velocity of 64 ft./sec. Assuming the acceleration as -32 ft./sec.², find the time at which the body reaches its greatest height. Find the distance covered during the third second.

16. The rate of change of the slope of a curve at any point (x,y) is $6ax + 2b$, and at $x = 1$ this rate of change equals 2. Find the equation of the curve if it is tangent to the line $5x - y + 7 = 0$ at the point $(-1,2)$ and passes through the point $(0,3)$.

17. A particle moves on a line so that its velocity at any time is given by $v = 3t^2 - 24t + 36$. Find the time at which the velocity is a maximum or a minimum and find that velocity. Find the distance covered by the particle between the instants when it comes to rest.

31. Rate of Change of Area.

Let the equation of the curve in Figure 21 be $y = f(x)$, where the curve is continuous, is rising and lies above the x-axis in the interval from A to N. Consider the area S bounded by the curve, the x-axis and the ordinates AB and MP. We shall assume that the area S is generated by the ordinate y as it moves from the fixed position AB to any other position MP. At any position of the ordinate y, as at MP, give to x an increment Δx. Let the corresponding increments of y and S be Δy and ΔS, respectively. In the figure $RQ = \Delta y$ and
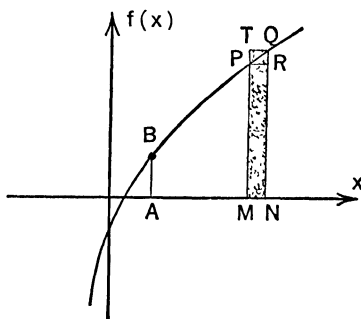
FIG. 21

$$\Delta S = \text{Area } MNQP.$$

From the figure it is obvious that

$$\text{Rectangle } MNRP < \Delta S < \text{Rectangle } MNQT,$$

or
$$y \cdot \Delta x < \Delta S < (y + \Delta y) \cdot \Delta x.$$

Dividing the inequality by Δx, we have

$$y < \frac{\Delta S}{\Delta x} < (y + \Delta y).$$

As Δx approaches zero, the ordinate NQ approaches coincidence with the ordinate MP, or $y + \Delta y$ approaches y as a limit. But since $\Delta S/\Delta x$ always lies between the two, it also must approach y as a limit. **Thus,**

$$\lim_{\Delta x \to 0} \frac{\Delta S}{\Delta x} = y,$$

or
$$\frac{dS}{dx} = f(x).$$

The rate of change of the area which is generated by an ordinate of a moving point on a curve, with respect to the abscissa, is equal to the ordinate of the point.

In the above discussion a continuous curve was assumed to be a rising curve above the x-axis in the interest of definiteness and simplicity. The argument may easily be modified to include the cases in which the curve falls steadily and in which it rises and falls alternately.

Now we shall assume a continuous curve $y = f(x)$ and consider the area between the curve and the x-axis from $x = a$ to $x = b$. We shall also assume the following conditions:

$$f'(x) = + \quad \text{for} \quad a < x < c, \quad f(x) = + \text{ for } x = a$$
$$f'(x) = 0 \quad \text{for} \quad x = c$$
$$f'(x) = - \quad \text{for} \quad c < x < b, \quad f(x) = - \text{ for } x = b,$$

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where $a < c < b$. Then the point $[c, f(c)]$ is a maximum point and the curve crosses the x-axis at the point $(x_1, 0)$ where $c < x_1, < b$. See Figure 22. Then, since

$$\frac{dS}{dx} = f(x),$$

the sign of the ordinates of points on the curve indicate the intervals over which the area between the curve and the x-axis is increasing or is decreasing and the point at which the area becomes maximum. Thus,
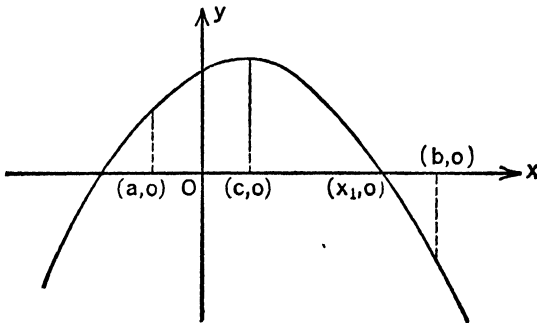
$$f(x) = + \quad \text{for } a < x < x_1, \, S \text{ is increasing.}$$

$$f(x) = 0 \quad \text{for } x = x_1, \, S \text{ is maximum.}$$

$$f(x) = - \quad \text{for } x_1 < x < b, \, S \text{ is decreasing.}$$

32. Area under a Curve.

It was proved in the last section that

$$\frac{dS}{dx} = f(x),$$

where S represents the area between a continuous curve $y = f(x)$ and the x-axis. From this we may express the area under the curve by means of an integral, thus

$$S = \int f(x) \, dx = F(x) + C.$$

This expression obtained for the area under a curve is a function of x and is indefinite because of the constant C. But also, the area under a curve is indefinite until it is bounded on the left and the right.

Let us consider the area under the curve $y = f(x)$ which is bounded on the left by the ordinate at $x = a$, and on the right by the ordinate at $x = b$. The variable area S will be small when x is only a little greater than a. In fact, the area S approaches zero as x approaches a. Therefore,

from above $\qquad 0 = F(a) + C, \quad \text{or} \quad C = -F(a).$

The variable area under the curve to the right of the line $x = a$ is

$$S = F(x) - F(a).$$

The area from $x = a$ to $x = b$ may now be found by evaluating this expression for $x = b$,

$$S = F(b) - F(a),$$

where $a < b$.

The area between a curve and the x-axis found in the manner indicated is positive if it is above the x-axis and is negative if it is below the x-axis. If an area between two ordinates is to be found which lies partly above the x-axis and partly below it, the two parts must be computed separately.

Otherwise, the algebraic sum of the two areas is found rather than the required arithmetic sum.

In Section 15 it was shown that the geometric interpretation of the derivative of a function is the slope of the curve. In an analogous manner, we may regard the area under a curve as the geometrical representation of the integral of a function. Thus, $dS = f(x)\,dx$ is the differential of the function $S = F(x) + C$ and the geometric interpretation of the integral is the area under the curve $y = f(x)$. It is for this reason that considerable emphasis is placed on the calculation of areas under curves by integration.

In order that this point may be illustrated still further, let us find the area under the curve

$$y = 4x - x^2$$

between the lines $x = 1$ and $x = 3$. The area under the curve is

$$S = \int (4x - x^2)\,dx = 2x^2 - \frac{x^3}{3} + C.$$

And the area between the two given lines is

$$S = F(3) - F(1) = (18 - 9) - \left(2 - \frac{1}{3}\right) = \frac{22}{3}\ \text{sq. units.}$$

Now suppose that we wish to find the distance covered by a particle moving on a line between the times $t = 1$ and $t = 3$, where the velocity at any time is given by

$$v = 4t - t^2,$$

and where t is in seconds and v is in feet per second. The distance of the particle from a fixed point at any time is

$$s = \int (4t - t^2)\,dt = 2t^2 - \frac{t^3}{3} + C.$$

And the distance covered between the specified times is

$$s = F(3) - F(1) = \frac{22}{3}\ \text{ft.}$$

If the equation $v = 4t - t^2$ is drawn on a pair of t- and v-axes, the same curve is obtained as from the equation above in x and y. Moreover, the areas under these curves between the given limits are the same. However, the area under the latter curve is the geometrical representation of the linear distance 22/3 ft. which is covered by the particle during the time specified.

To find the area between the curve $f(x) = x^3 - 3x^2 + 2x$ and the x-axis, it is necessary to know that the curve intersects the axis at the points $x = 0$, $x = 1$ and $x = 2$. The graph of the function shows that the first portion of the area lies above the x-axis and the latter, below it. The solution is carried out as follows:

$$S = F(x) = \int f(x)\,dx = \frac{x^4}{4} - x^3 + x^2 + C.$$

$$S = [F(1) - F(0)] - [F(2) - F(1)] = \tfrac{1}{2}.$$

Exercise 18

GROUP A

1. Find the area under $f(x) = 2x$ from $x = 1$ to $x = 4$. Verify the result by geometric methods.
2. Find the area under $f(x) = x^2 + 2$ from $x = 0$ to $x = 3$.
3. Find the area between the curve $f(x) = 4x - x^2$ and the x-axis.
4. Find the area between the curve $f(x) = 16 - x^2$ and the x-axis.
5. Find the area bounded by the curve $f(x) = x^3 - 6x^2 + 9x$ and the x-axis
6. Find the area bounded by the curve $f(x) = x^2 - 6x + 12$, the x-axis, the y-axis and the minimum ordinate.
7. Find the area bounded by the coordinate axes and $3x - 5y - 15 = 0$. Verify the result by geometric methods.
8. Find the area bounded by the curve $f(x) = x^2 - 8x$ and the x-axis.
9. Find the area bounded by the curve $f(x) = x^3 - 3x^2 + 2x$ and the x-axis.
10. Find the area bounded by the curve $f(x) = 2x^3 - 9x^2 + 12x$, the x-axis and the maximum and the minimum ordinates.

GROUP B

11. Find the area between the curve $y = 4x - x^2$ and $y = 2x$.
12. Find the area bounded by the curve $f(x) = x^3 + 3x^2 + 2$, the x-axis and the maximum and the minimum ordinates.
13. Find the area bounded by the curves $f(x) = x^2$ and $f(x) = 18 - x^2$.
14. Find the area of the segment of the parabola $y^2 = 4x$ cut off by the chord through the focus perpendicular to the axis
15. Find the area of the segment of the parabola $x^2 = 16y$ cut off by the chord through the focus perpendicular to the axis.
16. Find the area bounded by the curve $x + y^2 - 2y = 0$ and the y-axis.
17. A curve passes through the point $(2,0)$ and the slope at any point is $3x^2 - 2x - 1$. Find the equation of the curve and draw it.
18. The x-intercepts of a curve are -2 and 4. Find the equation of the curve and draw it if the rate of change of the slope is 2
19. The velocity of a particle moving on a line at any time is $v = t^2 - 5t + 4$. During what time is s decreasing? Find the distance covered during that time.
20. The velocity of a particle moving on a line at any time is $v = 6t - 5 - t^2$. During what time is s increasing? Find the distance covered during that time.

GROUP C

21. Twelve hundred persons would buy tickets to a moving picture show if they cost 20 cents each. Each cent added to the cost would deter 20 persons from buying tickets. Find the price the manager should charge per ticket in order to receive the greatest gross income.

22. Find the approximate error in the volume and the surface area of a cube of edge 6 ins., if an error of 0.02 in. is made in measuring the edge.

23. Find how exactly the diameter of a circle must be measured in order that the area computed from it shall be correct to within one per cent.

24. Write the expression for the area of a square inscribed in a circle of radius a. Find approximately the area remaining, if such a square is cut out of a circle of radius 3.99 ins.

25. Find the area of that portion of the segment of the curve whose equation is $y = 6 + x - x^2$ cut off by the line through the points $(-1,4)$ and $(3,0)$.

CHAPTER V

33. Classification of Functions.

All functions are divided into two large classes, *algebraic functions* and *transcendental functions.*

In this chapter we are concerned with the differentiation of algebraic functions. Prior to this, we have considered the differentiation and integration of a limited class of such functions, namely, rational integral functions or polynomials. It is now our task to extend these studies to other classes of functions.

Other algebraic functions are *rational fractional functions* and *irrational functions.* Such functions may be obtained from polynomials by multiplication, division, raising to powers and extracting roots. Thus,

$$(x - 2)^3(x^2 + 2) \quad \text{and} \quad (x^3 - 3x^2 + x)^4$$

are rational integral functions, while

$$\frac{x - 1}{x + 2} \quad \text{and} \quad \frac{x^2 - 3x + 11}{(x - 1)^3}$$

are rational fractional functions. The functions

$$\sqrt{x - 2} \quad \text{and} \quad (a^2 - x^2)^{2/3}$$

are illustrations of irrational functions.

Examples of transcendental functions are found in the trigonometric and inverse trigonometric functions and in the exponential and logarithmic functions. The differentiation and integration of such functions are presented in later chapters.

Single and Multi-valued Functions. A *single-valued function* is one in which there is one, and only one, value of the function corresponding to each value of the independent variable. A function is called *multi-valued* if to some value of the independent variable, there corresponds two or more values of the function.

In dealing with multi-valued functions it is customary to group the values so as to form two or more one-valued functions. These one-valued

functions are called *branches* of the function and the corresponding parts of the curve are called branches of the whole curve. It shall be our practice to attend to a single branch of the function or the curve, or to deal with the branches separately. In this way, we may limit our study to that of a one-valued function. In illustration,

$$y^2 = x + 1$$

is a two-valued function of which the branches are

$$y = \sqrt{x + 1} \quad \text{and} \quad y = -\sqrt{x + 1},$$

and the two branches of the curve are the upper and the lower half of the parabola, respectively.

The formulas for the differentiation of algebraic functions which are derived in this chapter are direct consequences of the definition of the derivative given in Section 13 and are valid for all functions which are continuous and one-valued and which possess derivatives.

34. Differentiation Formulas.

In Section 14 the following formulas were derived:

$$(1) \qquad\qquad \frac{d}{dx}(a) = 0,$$

where a denotes any constant.

$$(2) \qquad\qquad \frac{d}{dx}(x^n) = nx^{n-1},$$

where n is a rational number, proved for n an integer only.

$$(3) \qquad\qquad \frac{d}{dx}(au) = a\frac{du}{dx},$$

where u represents a function of x which can be differentiated.

$$(4) \qquad\qquad \frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx},$$

where u and v are functions of x which can be differentiated.

In the next section the following formulas are derived:

$$(5) \qquad\qquad \frac{d}{dx}(u \cdot v) = u\frac{dv}{dx} + v\frac{du}{dx},$$

the formula for the differentiation of the product of two functions of x.

$$(6) \qquad \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}, \qquad v \neq 0,$$

the formula for the differentiation of the quotient of two functions of x.

$$(7) \qquad \frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx},$$

the formula for the differentiation of a function of x to a rational power.

35. Derivations of Differentiation Formulas.

Let $$y = u \cdot v,$$

where u and v represent two functions of x which can be differentiated. Give to x an increment Δx, and let the corresponding increments of u, v and y be Δu, Δv and Δy, respectively. Then

$$y + \Delta y = (u + \Delta u)(v + \Delta v),$$

$$\Delta y = u\,\Delta v + v\,\Delta u + \Delta u\,\Delta v,$$

and $$\frac{\Delta y}{\Delta x} = u\frac{\Delta v}{\Delta x} + v\frac{\Delta u}{\Delta x} + \frac{\Delta u}{\Delta x}\Delta v.$$

Since $$\lim_{\Delta x \to 0}\frac{\Delta u}{\Delta x} = \frac{du}{dx}, \quad \lim_{\Delta x \to 0}\frac{\Delta v}{\Delta x} = \frac{dv}{dx}$$

and $$\lim_{\Delta x \to 0}\Delta v = 0, \text{ we have}$$

$$(5) \qquad \frac{d}{dx}(u \cdot v) = u\frac{dv}{dx} + v\frac{du}{dx}$$

Derivative of a Product. *The derivative of the product of two functions is equal to the first times the derivative of the second plus the second times the derivative of the first.*

Let $$y = \frac{u}{v}, \qquad v \neq 0,$$

where u and v are two functions of x which can be differentiated. Give to x an increment Δx, and let the corresponding increments of u, v and y be

Δu, Δv and Δy, respectively.   Then

$$y + \Delta y = \frac{u + \Delta u}{v + \Delta v}, \quad \Delta y = \frac{v\,\Delta u - u\,\Delta v}{v(v + \Delta v)},$$

$$\frac{\Delta y}{\Delta x} = \frac{v\,\dfrac{\Delta u}{\Delta x} - u\,\dfrac{\Delta v}{\Delta x}}{v(v + \Delta v)}, \quad \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{v\,\lim\limits_{\Delta x \to 0}\dfrac{\Delta u}{\Delta x} - u\,\lim\limits_{\Delta x \to 0}\dfrac{\Delta v}{\Delta x}}{v\,\lim\limits_{\Delta x \to 0}(v + \Delta v)}.$$

In taking the limit as Δx approaches zero, Δu and Δv also approach zero.

$$(6) \qquad\qquad \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\,\dfrac{du}{dx} - u\,\dfrac{dv}{dx}}{v^2}, \qquad v \neq 0.$$

Derivative of a Quotient. *The derivative of the quotient of two functions is equal to the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator.*

Let $\qquad\qquad\qquad\qquad y = u^n$,

where u is a function of x which can be differentiated and where n is a positive integer. Give to x an increment Δx, and let the corresponding increments of u and y be Δu and Δy, respectively.   Then

$$y + \Delta y = (u + \Delta u)^n.$$

Expanding the right-hand member of the latter equation by the binomial theorem and subtracting u^n from the result,

$$\Delta y = nu^{n-1}\;\Delta u + \frac{n(n-1)}{1 \cdot 2}\,u^{n-2}\;\overline{\Delta u}^2 + \cdots + \overline{\Delta u}^n.$$

Dividing by Δx,

$$\frac{\Delta y}{\Delta x} = nu^{n-1}\;\frac{\Delta u}{\Delta x} + \frac{n(n-1)}{1 \cdot 2}\,u^{n-2}\frac{\Delta u}{\Delta x}\Delta u + \cdots + \frac{\Delta u}{\Delta x}\overline{\Delta u}^{n-1}$$

Taking the limit as Δx approaches zero, we have

$$(7) \qquad\qquad \frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}.$$

Derivative of a Function to the nth Power. *The derivative of a function to the nth power is equal to n times the product of the function to the $(n - 1)$th power and the derivative of the function.*

The application of formulas (5) and (7) is illustrated by the differentiation of the function

$$f(x) = (x^2 + 1)(2x - x^2)^2.$$

$$f'(x) = (x^2 + 1)\frac{d}{dx}(2x - x^2)^2 + (2x - x^2)^2\frac{d}{dx}(x^2 + 1),$$

$$= (x^2 + 1)\,2\,(2x - x^2)\frac{d}{dx}(2x - x^2) + (2x - x^2)^2\,2x,$$

$$= 2(2x - x^2)\,[(x^2 + 1)\,(2 - 2x) + x(2x - x^2)],$$

$$= 2(2x - x^2)(2 - 2x + 4x^2 - 3x^3).$$

The application of formulas (6) and (7) is illustrated by the differentiation of the function

$$f(x) = \frac{(x - 2)^3}{x + 1}.$$

$$f'(x) = \frac{(x + 1)\dfrac{d}{dx}(x - 2)^3 - (x - 2)^3\dfrac{d}{dx}(x + 1)}{(x + 1)^2} \cdot$$

$$= \frac{(x + 1)\,3(x - 2)^2\dfrac{d}{dx}(x - 2) - (x - 2)^3}{(x + 1)^2}$$

$$= \frac{(x - 2)^2\,[3(x + 1) - (x - 2)]}{(x + 1)^2} = \frac{(x - 2)^2(2x + 5)}{(x + 1)^2} \cdot$$

In the derivation of formula (7), n was restricted to positive integral values. Now we shall show that this formula is also valid for negative integral values of n and for positive and negative rational fractional values of n.

Let

$$n = \frac{p}{q},$$

where p and q are positive integers which are prime to each other. Then

$$y = u^{p/q}.$$

Raise both sides of the equation to the qth power

$$y^q = u^p,$$

in which we have two functions of x, equal for all values of x. Taking the

derivative of both sides of the equation by the use of formula (7),

$$qy^{q-1}\frac{dy}{dx} = pu^{p-1}\frac{du}{dx},$$

$$\frac{dy}{dx} = \frac{p}{q}\frac{u^{p-1}}{y^{q-1}}\frac{du}{dx},$$

or

$$\frac{dy}{dx} = \frac{p}{q}u^{(p/q)-1}\frac{du}{dx}.$$

Upon replacing p/q by n, we have formula (7).

Again, let $\qquad n = -m,$

where m is a positive rational number. Then

$$y = \frac{1}{u^m}.$$

Applying formulas (6) and (7),

$$\frac{dy}{dx} = \frac{-mu^{m-1}\dfrac{du}{dx}}{u^{2m}},$$

$$\frac{dy}{dx} = -mu^{-m-1}\frac{du}{dx}.$$

Upon replacing $-m$ by n, we have formula (7).

Exercise 19

GROUP A

Differentiate each of the following functions.

1. $f(x) = (2x - 1)(x^3 + 1)$.

2. $f(x) = (2 - 3x)(x^2 + 1)$.

3. $f(x) = \dfrac{x}{x - 2}.$

4. $f(x) = \dfrac{2x - 1}{1 - 3x}.$

5. $f(x) = (3x^2 - 2)^3$.

6. $f(x) = \sqrt{x^2 - 3x + 2}.$

7. $y = (x^2 - 2)(2x^2 - 2x + 3)$.

8. $y = \sqrt{(x^2 - 1)^3}.$

9. $y = \dfrac{1 - x^2}{1 + x^2}.$

10. $y = \dfrac{1}{(1 - x)^2}.$

11. Given $y = x(x - 1)^2$. Find the coordinates of any maximum, minimum and inflection points and draw the curve.

12. Given $y = 2/(x - 1)^2$. Find the equation of the tangents at the points where $x = 2$ and $x = -1$.

13. Given $y = x\sqrt{4 - x^2}$. Find the rate of change of y with respect to x for $x = 1$ and $x = -1$.

14. Given $y = x(3 - x)^3$. Find the coordinates of the critical points and the interval in which the curve is concave upward.

15. Given $y = x(1 - x)^4$. Draw the curve and the first and second derived curves.

GROUP B

Differentiate each of the following functions.

16. $y = x\sqrt{x^2 - 2}$.

17. $y = \dfrac{x}{(2 - 3x)^2}$.

18. $y = (3 - 4x)\sqrt{3x - 2x^2}$.

19. $y = (2x - 1)^2(2x + 1)^3$.

20. $y = \dfrac{(x - 2)^2}{\sqrt{x^2 + 1}}$.

21. $f(x) = (x^2 + 2)\sqrt{(1 - x)^3}$.

22. $f(x) = \dfrac{\sqrt[3]{x^2 - 3x^3}}{x}$.

23. $y = \dfrac{\sqrt{3 - x}}{\sqrt{3 + x}}$.

24. $y = \sqrt{\dfrac{x}{1 - x^2}}$.

25. $f(t) = \dfrac{(1 + t)(1 - 2t)}{t}$.

26. A particle moves on a line so that its distance from a fixed point at any time is $s = t(t - 4)^3$. Draw the s, v and j curves and make an analysis of the motion.

27. Find the equations of the tangents to $y(x - 2) = x$ at the origin and at $x = 3$.

28. Given $xy = x^2 + 1$. Find the intervals for which the curve is concave upward and downward

29. Rectangles are inscribed in a semicircle of radius $\sqrt{5}$, with one side along the diameter. Find the dimensions of the rectangle whose perimeter has a critical value and show whether it is a maximum or a minimum.

30. Given $y = 4x^2(1 - x)^2$. Find coordinates of any maximum, minimum and inflection points and give the intervals in which the curve is concave upward and downward. Draw the curve.

GROUP C

Differentiate each of the following functions.

31. $f(t) = t(t^3 - 2)(t^2 + 1)$.

32. $f(z) = (a^{2/3} - x^{2/3})^3$.

33. $f(z) = \dfrac{z}{\sqrt{(b^2 - z^2)^3}}$.

34. $f(y) = y^2\sqrt{c^2 - y^2}$.

35. $f(y) = \dfrac{y\sqrt{y - b}}{y + b}$.

36. $v = \dfrac{w(a - w)^2}{a + w}$.

37. $z = \dfrac{b}{x^2(x - a)^2}$.

38. $y = \dfrac{ax(a - x)^3}{b(a + x)^2}$.

39. $y = \sqrt[3]{\dfrac{x^3 + a}{x^3 - a}}$.

40. $f(x) = \dfrac{\sqrt[3]{x - 4x^4}}{x^2}$.

41. Three towns A, B and C are so situated that they form an isosceles triangle with the base AB of 12 miles. The altitude through C is 9 miles. Three roads $AD = BD$ and CD are to be built to a point D. Find the length CD so that the three roads have the least total length.

42. If $x = 4.002$ find the approximate values of $1/(x + 1)$ and $1/\sqrt{9 + x^2}$).

43. If $y = \sqrt{t}$ and $x = 1/(1 + t^2)$ find dy/dx.

44. If $y = (2t + 1)/t^2$ and $x = t$. Find the coordinates of the minimum point of the curve.

45. A particle moves on a line so that its distance from a fixed point at any time is $s = (2 - t)^2/t$. Disregarding negative values of t, find the time at which the velocity is zero and the acceleration at that time.

36. Implicit Algebraic Functions.

Consider an algebraic equation containing the variables x and y. The symbol for the representation of such an equation may be

$$F(x,y) = 0.$$

The existence of such a relationship between x and y implies that y is a function of x. Under these circumstances, y is called an *implicit function of x*. Likewise, x is an implicit function of y. In the first instance, x is thought of as the independent variable. If the equation is solved for y in terms of x, obtaining

$$y = f(x),$$

y becomes an *explicit function of x*, since y is expressed explicitly in terms of x. Were the equation solved for x, x would be an explicit function of y.

In the equation

$$2xy + x + y^2 - 3 = 0,$$

each variable is an implicit function of the other. Upon solving for y,

$$y = -x \pm \sqrt{x^2 - x + 3},$$

in which y is an explicit function of x.

When working with equations in two variables, frequently it is not possible, nor desirable, to change the equation to explicit form.

37. Implicit Differentiation.

Given a function defined implicitly,

$$F(x,y) = 0,$$

in which it is desired to find the derivative of y with respect to x. To do this it is unnecessary to express y as an explicit function of x. Each term of the equation is differentiated with respect to x. Having in mind that y is a function of x, by formula (7) the derivative of y^n with respect to x is

$$ny^{n-1}\frac{dy}{dx}.$$

In general, after the differentiations are carried out, an equation in terms of x, y and dy/dx is obtained. This equation may be solved for dy/dx in terms of x and y.

If a product or a quotient term appears in the equation, formula (5) or (6) must be applied. For example, if

$$xy = x + 2,$$

we apply (5) as follows:

$$x\frac{dy}{dx} + y = 1$$

and

$$\frac{dy}{dx} = \frac{1-y}{x}.$$

Or again, if

$$\frac{x}{y} = x + 2,$$

we apply (6) as follows:

$$\frac{y - x\dfrac{dy}{dx}}{y^2} = 1$$

and

$$\frac{dy}{dx} = \frac{y(1-y)}{x}.$$

An equivalent result in different form, might have been obtained for the latter differentiation had we cleared the equation of fractions and applied (5) as in the first equation.

Let it be required to find the slope of the curve

$$xy^2 + 2y - 4 = 0$$

at the point (2,1). Differentiating the function implicitly,

$$y^2 + 2xy\frac{dy}{dx} + 2\frac{dy}{dx} = 0,$$

or

$$\frac{dy}{dx} = \frac{-y^2}{2xy + 2}.$$

Evaluating the slope at the given point, we have

$$\frac{dy}{dx} = -\frac{1}{6}.$$

It is possible also to use differentials in implicit differentiation.

From the last equation

$$y^2dx + 2xy\,dy + 2\,dy = 0, \quad \text{or} \quad dy = \frac{-y^2}{2xy + 2}\,dx.$$

From this, the derivative is the same function as before.

Exercise 20

GROUP A

In each of the following equations express y as an explicit function of x.

 1. $x + 3y - 7 = 0$.
 2. $x^2 + y^2 = 4$.
 3. $xy - 4x + 2y = 0$.
 4. $x^2 - y^2 + 2y = 0$.

Find the derivative of y with respect to x in each of the following equations.

 5. $xy - 4 = 0$.
 6. $x^2 + y^2 - 4x + 2y - 1 = 0$.
 7. $x^2 - xy + y^2 + 2x = 6$.
 8. $y^2 + 2x - 4y - 4 = 0$.
 9. $x^2 + xy - y^2 = 2$.
 10. $2xy + y^2 = 6$.

Find the equation of the tangent to each of the following curves at the specified points.

 11. $x^2 + y^2 = 25$ at $(3,4)$ and at $(-3,4)$.
 12. $y^2 - 8x = 0$ at $(2,-4)$ and at $(2,4)$.
 13. $y^2 + x + 2y - 9 = 0$ at $(1,2)$.

14. Find the angle between the tangents to $x^2 + y^2 = 25$ at the points $(3,4)$ and $(4,3)$.

15. Given the equation $x^2y^2 - x^2 - 2 = 0$. Find dy/dx both by implicit and explicit differentiation and show that the results are equal.

GROUP B

Find the derivative of y with respect to x in each of the following equations.

 16. $y^2 + 2ax - a^2 = 0$.
 17. $b^2x^2 + a^2y^2 = a^2b^2$.
 18. $b^2x^2 - a^2y^2 = a^2b^2$.
 19. $x^{1/2} + y^{1/2} = a^{1/2}$.
 20. $x^{2/3} + y^{2/3} = b^{2/3}$.

Find the equation of the tangent to each of the following curves at the specified points.

 21. $y^2 + x - y - 9 = 0$ at $(7,2)$.
 22. $x^2 + y^2 - 4x + 6y - 12 = 0$ at $x = 5$. Show that the tangent is perpendicular to the radius of the circle at the given point.
 23. $2x^3 - y^2 = 0$ at $x = 2$.
 24. $y^2 - 4px = 0$ at (x_1, y_1).
 25. $x^2 + y^2 = a^2$ at (x_1, y_1).

Find the angle at which the following curves intersect.

26. $3y^2 - 16x = 0$ and $4x^2 - 9y = 0$.

27. $y^2 - 8x = 0$ and $x^2 + y^2 - 5y = 0$.

28. $x^2 + y^2 - 4 = 0$ and $y^2 - 2x - 4 = 0$.

Find the derivative of y with respect to x in each of the following equations.

29. $xy + ax + by + c = 0$.

30. $x^2y^2 + x^2 + y^2 + a^2 = 0$.

31. $xy^2 - y^3 + ax + by = 0$.

32. $(x + y)^2(x - y) = a$.

33. $y^2 = x + \sqrt{x^2 + y^2}$.

38. Successive Implicit Differentiation.

In Section 20 the first, the second and higher derivatives of $y = f(x)$ with respect to x were represented by $f'(x)$, $f''(x)$, $f'''(x)$, $f^{\text{iv}}(x)$, etc. The continued differentiation of a function with respect to its independent variable gives rise to functions which are known as the *successive derivatives* of the function. The *order* of the successive derivative is the number of times the given function has been differentiated.

The successive derivatives of a function of two variables with respect to one of them can be found by *continued implicit* differentiation. If $F(x,y) = 0$ is differentiated successively with respect to x, it is convenient to use the symbols

$$\frac{dy}{dx}, \quad \frac{d^2y}{dx^2}, \quad \frac{d^3y}{dx^3}, \quad \frac{d^4y}{dx^4}, \quad \text{etc.,}$$

for the successive derivatives of y with respect to x.

For example, let us find the third derivative of y with respect to x from the equation

$$x^3 + y^3 = a^3.$$

Differentiating implicitly,

$$3x^2 + 3y^2 \frac{dy}{dx} = 0, \quad \text{or} \quad x^2 + y^2 \frac{dy}{dx} = 0.$$

From this

$$\frac{dy}{dx} = -\frac{x^2}{y^2}.$$

If we use the second equation and differentiate,

$$2x + 2y \left(\frac{dy}{dx}\right)^2 + y^2 \frac{d^2y}{dx^2} = 0,$$

or

$$2x + \frac{2x^4}{y^3} + y^2 \frac{d^2y}{dx^2} = 0$$

by substituting the expression for the first derivative found above. Solving for the second derivative,

$$\frac{d^2y}{dx^2} = -(x^3 + y^3)\frac{2x}{y^5} = -\frac{2a^3x}{y^5}.$$

From this result, the third derivative is found to be

$$\frac{d^3y}{dx^3} = -\frac{2a^3}{y^{10}}\left(y^5 - 5xy^4\frac{dy}{dx}\right)$$

$$= -\frac{2a^3(y^3 + 5x^3)}{y^8},$$

When working with parametric equations

$$y = f_1(t), \quad x = f_2(t),$$

it often becomes necessary to find the derivatives of y with respect to x of higher order than the first. In Section 26 it was proved that

$$\frac{dy}{dx} = \frac{f_1'(t)}{f_2'(t)},$$

where the primes indicate differentiation with respect to t. In general, the result is a function of t. In order to obtain higher derivatives, this result must be *differentiated with respect to x*.

Thus, if we have the parametric equations

$$y = t^2 + 1, \quad x = t^2 + 2t,$$

$$\frac{dy}{dx} = \frac{t}{t + 1}.$$

Differentiating with respect to x,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{t}{t + 1}\right) = \frac{1}{(t + 1)^2}\frac{dt}{dx}.$$

From the second given equation, we have

$$1 = 2(t + 1)\frac{dt}{dx}.$$

Hence,
$$\frac{d^2y}{dx^2} = \frac{1}{2(t + 1)^3}.$$

Algebraic Functions

Exercise 21

GROUP A

Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ for each of the following equations.

1. $y^2 = 4ax$.
2. $y^3 = 4x$.
3. $xy = a^2$.
4. $y^3 = 4ax^2$.
5. $x^2 + y^2 = a^2$.
6. $x^2 - y^2 = a^2$.

7. $xy + 3x + 2y = a$.
8. $xy^2 = a$.
9. $x^2y^2 = a$.
10. $x^2 + xy - y^2 = 0$.
11. $y = t^3, x = t^2$.
12. $y = 3t^2, x = 2 - t$.

GROUP B

13. Find the first and second derivative of y with respect to x if $\sqrt{x} + \sqrt{y} = \sqrt{a}$.

14. Find $\dfrac{d^3y}{dx^3}$ if $y^2 = ar^3$

15. Find $\dfrac{d^3y}{dx^3}$ if $x^2 + y^2 = a^2$.

16. Find $\dfrac{dy}{dx}$ if $x^2 + 2x\sqrt{y} - x - y^2 + 2 = 0$.

17. Find the derivative of y with respect to x if $\dfrac{y}{x} + \sqrt[3]{\dfrac{x}{y}} = 1$.

18. Find $\dfrac{dy}{dx}, \dfrac{d^2y}{dx^2}$ and $\dfrac{d^3y}{dx^3}$ if $x^2 + y^2 - 2ax - 2by = 0$.

19. Find $\dfrac{d^4y}{dx^4}$ if $y^2 = x + y$.

20. Find the coordinates of the critical point of the curve $4x^2y - 8xy^2 + 3y^3 = 8$ and show it to be maximum or minimum by the use of the second derivative.

21. Find the coordinates of the maximum and minimum points of the curve

$$x^2 - 2xy - 4y^2 + 5 = 0$$

and show the maxima-minima test.

22. Find the dimensions and the area of the maximum rectangle which can be inscribed in the curve $3x^{2/3} + y^{2/3} = 24$.

Find and classify any critical points of each of the following curves.

23. $y = \dfrac{12t - t^3}{8}, x = \dfrac{t}{2}$.

24. $y = t^2 - 8t, x = 2 - t$.

25. $4x^2y - 8xy^2 + 5y^3 = 1$.

26. $f(x) = x(6 + x)^2(6 - x)^3$.

27. Find the dimensions of the maximum rectangle which can be inscribed in the curve $x^{2/3} + y^{4/3} = 24$.

39. Applications of Implicit Differentiation to Maxima and Minima.

The implicit differentiation of functions makes it possible for us to simplify the method of finding the maxima and minima of functions as presented in Chapter II. If the function whose maximum or minimum

value is sought is expressed as a function of two variables, the data of the problem must enable one to relate these two variables by means of a second equation. Both equations may then be differentiated implicitly and an equation formed between the two variables which gives a solution. This method of approach can best be presented by means of an illustration.

To find the dimensions of the maximum rectangle which can be inscribed in the ellipse $x^2 + 4y^2 = 4$, we proceed as follows:

The area of any inscribed rectangle is

$$S = 4xy,$$

and since a critical value of S is desired,

$$\frac{dS}{dx} = 4\left(x\,\frac{dy}{dx} + y\right) = 0,$$

or

$$\frac{dy}{dx} = -\frac{y}{x}.$$

From the equation of the ellipse,

$$x + 4y\,\frac{dy}{dx} = 0 \quad\text{and}\quad \frac{dy}{dx} = -\frac{x}{4y}.$$

Equating the two values of the derivative,

$$-\frac{y}{x} = -\frac{x}{4y}, \quad x = 2y.$$

Thus, the maximum rectangle has a base equal to twice its altitude.

The same result is obtained if we substitute $dy/dx = -(x/4y)$, from the differentiation of the equation of the ellipse, in the equation $dS/dx = 0$, obtaining

$$\frac{dS}{dx} = 4\left(-\frac{x^2}{4y} + y\right) = \frac{4y^2 - x^2}{y} = 0.$$

As before, $x^2 = 4y^2$, or $x = \pm 2y$. The solution $x = -2y$ has no interpretation in the problem under consideration.

In order to make the second derivative test,

$$\frac{d^2S}{dx^2} = \left.\frac{y\left(8y\,\dfrac{dy}{dx} - 2x\right) - (4y^2 - x^2)\,\dfrac{dy}{dx}}{y^2}\right]^{x=2y}$$

$$= \frac{y(-4y - 4y) - 0}{y^2} = -8,$$

which, being negative, shows that $x = 2y$ gives S maximum. The dimensions of the rectangle are found to be

$$2\sqrt{2} \text{ by } \sqrt{2},$$

by solving this equation simultaneously with the equation of the ellipse, obtaining the solution $x = \sqrt{2}$ and $y = \sqrt{2}/2$.

Exercise 22

GROUP A

1. Find the proportions of the maximum isosceles triangle of given perimeter.
2. Find the most economical proportions for a conical funnel of given volume.
3. Find the minimum distance from $(0,3)$ to the curve $y^2 = 4x$.
4. The length of the hypotenuse of a right triangle is a. Find the dimensions if the area is maximum.
5. The strength of a rectangular beam varies as the product of its breadth and the cube of its depth. Find the dimensions of the end of a beam of maximum strength which can be cut from a given cylindrical log.
6. A pyramid with a square base is to have a maximum volume. Find the proportion of the dimensions for a fixed lateral surface.
7. A sector is to be cut from a given circular piece of tin. Find the proportions of the right circular cone which can be formed having a maximum volume.
8. Find the most economical proportions for a cylindrical can for a given volume if the material used for the bottom is twice as expensive per unit area as that used for the sides and the top.
9. From 500 sq. ft. of tin find the proportions of the box with no top and a square base which can be made having the maximum volume. No allowance is to be made for waste or for the dimensions of the material.
10. A post 10 ft. high stands 20 ft. from an electric light pole 15 ft. high. A guy wire is to be stretched from the top of each to a stake between them. Find the position of the stake in order that the length of the wire be least.

GROUP B

11. Find the dimensions of the largest rectangle inscribed in the ellipse $b^2x^2 + a^2y^2 = a^2b^2$.
12. Find the dimensions of the largest rectangle with two vertices on the curve $2x^2 - 3y^2 = 6$ and two vertices on the line $x - 5 = 0$.
13. Find the area of the maximum isosceles triangle inscribed in $b^2x^2 + a^2y^2 = a^2b^2$ with its base parallel to the major axis, assuming $a > b$.
14. The two sides of a long strip of tin of given width are to be bent up to form an open trough. If the sides are bent to form an angle of 60° with the horizontal, find the dimensions in order that the trough have a maximum carrying capacity.
15. Given $y(x^2 + 4) = 8$. Find the coordinates of the critical and inflection points of the curve.
16. Given $y(x^2 + 1) = x$. Find the coordinates of the maximum, minimum and the inflection points of the curve.

17. A tank is to be built in the form of a half cylinder with no top and with semicircular ends. If the material for the ends costs twice as much per unit area as that for the convex surface, find the dimensions for the minimum cost for a given volume.

18. A power station is located on the edge of a cliff and it is 2 miles across to the cliff on the opposite side of the valley. It is desired to construct a high tension wire across the valley to a town which is 20 miles directly down the valley from the point opposite the power station. If it costs 100 dollars per mile to construct the cable across the valley and 60 dollars per mile along the cliff, find the most economical way to build the cable, assuming the wires in a horizontal plane.

19. A vessel is anchored 6 miles off shore. A man wishes to go to a point on shore 10 miles from the vessel. If he can row 2 mi. per hr. and walk 4 mi. per hr., find where he should land in order to reach his destination in the shortest possible time.

20. Find the dimensions of the isosceles triangle inscribed in $b^2x^2 + a^2y^2 = a^2b^2$ with its base parallel to the major axis such that the cone generated by rotating the triangle about its altitude has a maximum volume.

21. A rectangular sheet of paper is a inches wide. One corner is to be folded over so as to lie on the opposite edge of the sheet. Find the width of the part folded over in order that the length of the crease be minimum.

22. Regarding Problem 21, find the width of the part folded over in order that the area of that part be minimum.

40. Curve Tracing.

In many of the applications of the calculus it is essential to have precise knowledge concerning the position and the behavior of a curve which represents the function under consideration. The importance of being able to trace a curve from its equation, quickly and accurately, cannot be overemphasized. It is, therefore, pertinent that a few remarks be made on the elementary part of the subject of *curve tracing.*

The analysis of an equation preparatory to the tracing of the curve is here considered under two headings, the analytic study and the calculus study.

Analytic Study. It is often possible to trace a curve readily by an investigation of the equation by the methods of elementary analytic geometry. These methods of study, as applied to the simpler types of equations, are also most useful in obtaining information concerning the more difficult types of equations in which a calculus study is to be made.

The usual topics included in any analytic study of an equation are as follows:

Symmetry. A curve $F(x,y) = 0$ is symmetrical with respect to the x-axis, if the equation is unchanged by a substitution of $-y$ for y. The curve is symmetrical with respect to the y-axis, if the equation is unchanged by a substitution of $-x$ for x. Finally, a curve is symmetrical with respect

to the origin, if the equation is unchanged by a simultaneous substitution of $-y$ for y and $-x$ for x.

These symmetry relations may be extended to include a study of symmetry with respect to lines parallel to the coordinate axes. For example, consider the curves representing the equations,

$$A(x - a)^2 + By = 0, \quad B(y + b)^2 + Ax^2 + Dx = 0$$

and $\qquad A(x - a)^2 + B(y + b)^2 = C^2.$

The first is symmetrical with respect to the line $x - a = 0$, the second is symmetrical with respect to the line $y + b = 0$ and the third is symmetrical with respect to both lines and hence, symmetrical with respect to the point $(a, -b)$.

Intercepts. The x-intercepts of a curve $F(x,y) = 0$ are the real solutions of the equation $F(x,0) = 0$. Similarly, the y-intercepts of the curve are the real solutions of the equation $F(0,y) = 0$.

If the equation $F(x,0) = 0$ has a multiple root, use should be made of the calculus in order to determine whether the intercept is an inflection point or a maximum or a minimum point. A discussion of a more general case is given below under the calculus study of inflection points.

Extent. An equation $F(x,y) = 0$, should be solved for both x and y, if possible, obtaining

$$y = f(x) \quad \text{and} \quad x = g(y).$$

Those values of x for which $f(x)$ is real, define the intervals along the x-axis in which the curve lies. Those values of x for which $f(x)$ is imaginary, define the intervals along the x-axis in which there is no part of the curve. Similarly, the intervals along the y-axis may be studied for values of y giving $g(y)$ real and imaginary.

Horizontal and Vertical Asymptotes. In the equation $y = f(x)$, any value of x, as $x = a$, for which $f(a)$ becomes infinite, locates a vertical asymptote $x - a = 0$. If such is the case, the function $f(x)$ is discontinuous for $x = a$. In the equation $x = g(y)$, any value of y, as $y = b$, for which $g(b)$ becomes infinite, locates a horizontal asymptote $y - b = 0$.

In an application, such as finding the area under a curve, if the function involved has a discontinuity within the interval concerned, it is highly important to take that fact into consideration. In the same way, it is often necessary to take into consideration a horizontal asymptote in certain applications. Examples of some of these applications are to be found in Chapter IX.

The tracing of the curve corresponding to the equation

$$xy^2 + 2y - 4 = 0$$

is given as an illustration in which the methods of analytic geometry give sufficient information to draw the curve.

The solutions for y and for x are

$$y = \frac{-2 \pm 2\sqrt{1 + 4x}}{x}, \quad x = \frac{2(2 - y)}{y^2}.$$

There is no symmetry with respect to the coordinate axes. The y-intercept is 2. For values of x less than $-\frac{1}{4}$, y is imaginary. For values of x greater than $-\frac{1}{4}$, y is real. Both the x-axis and the y-axis are asymptotes. The curve is drawn in Figure 23 by using this information and by plotting several points of the curve.
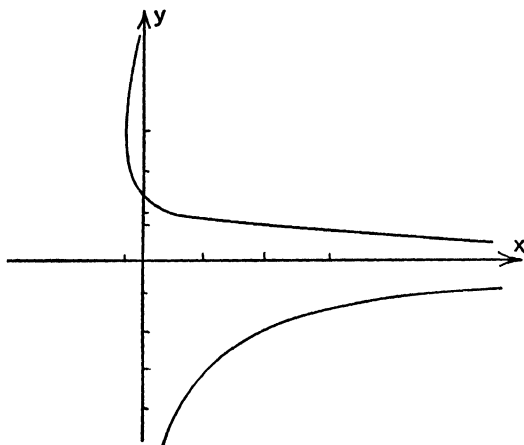

Fig. 23

It is not considered necessary here to give an analytic method for finding oblique asymptotes. However, it is sometimes advisable to make an investigation of a function for large values of the variable. Such a study may or may not lead to the discovery of an oblique asymptote. As an illustration in which such an asymptote exists, consider the equation,

$$x^2 - xy - 1 = 0.$$

The solutions for y and for x are

$$y = x - \frac{1}{x}, \quad x = \frac{y}{2} \pm \frac{1}{2}\sqrt{y^2 + 4}.$$

There is no symmetry with respect to the coordinate axes. The x-intercepts are -1 and 1. There is no interval along the x-axis for which the first function is imaginary, nor along the y-axis for which the second function is imaginary. The y-axis is a vertical asymptote. In the solution of the given equation for y, it is to be observed that as x becomes infinite, $y - x$ approaches zero, since $1/x$ approaches zero. Therefore, the line $y - x = 0$ is an oblique asymptote. The curve is the hyperbola drawn in Figure 24.
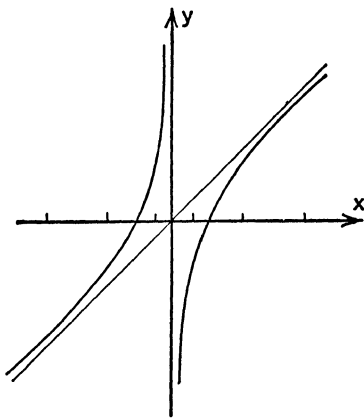
Calculus Study. The analysis of an equation preparatory to tracing the curve by the methods of the calculus has been applied to polynomials which are continuous single-valued functions. These methods may now be applied to other functions which may or may not be single-valued and which may or may not be continuous for all values of the variable. A summary of the methods of the calculus as applied to curve tracing is given as follows:

Fig. 24

Critical Points. The abscissas of the critical points of a curve $y = f(x)$ are the real solutions of the equation $f'(x) = 0$. At these points the tangents to the curve are horizontal. The criteria for maxima and minima are applied and the points are found to be maximum points, minimum points or neither.

Inflection Points. If $y = f(x)$, the real roots of the equation $f''(x) = 0$ are the abscissas of inflection points of the curve only in case the second derivative changes sign as x varies through those values.

The first and the second derivatives of the function $f(x) = (x - 1)^3 + 2$ are

$$f'(x) = 3(x - 1)^2 \quad \text{and} \quad f''(x) = 6(x - 1).$$

The critical value of the function is $f(1) = 2$. The second derivative test of the critical value fails, since $f''(1) = 0$. The first derivative does not change sign as x increases through $x = 1$ and the second derivative does change sign. Hence, the point $(1,2)$ is a critical inflection point of the curve.

The first and second derivatives of the function $f(x) = (x - 1)^4 + 2$ are

$$f'(x) = 4(x - 1)^3 \quad \text{and} \quad f''(x) = 12(x - 1)^2.$$

As in the function above, the critical value of this function is $f(1) = 2$ and the second derivative test fails, since $f''(1) = 0$. In this case, the first derivative changes from negative values to positive values as x increases through $x = 1$ and the second derivative does not change sign. Hence, the point (1,2) is a minimum point of the curve.

Concavity. If $y = f(x)$, the range of x for which $f''(x) < 0$, defines an interval over which the curve is concave downward. The range of x for which $f''(x) > 0$, defines an interval over which the curve is concave upward.

Suppose that an equation $F(x,y) = 0$ so that $y = f(x)$ is not a polynomial. The curves representing such equations often possess points which merit special attention. The power of the calculus is shown in the following investigation of these particular points.

Singular Points. Suppose that $F(x,y) = 0$ is an equation such that by implicit differentiation, the derivative of y with respect to x is a function of x and y, taking the form

$$\frac{dy}{dx} = \frac{g(x,y)}{q(x,y)}.$$

If there are values of x and y, as (x_1,y_1), which satisfy the three equations

$$F(x,y) = 0, \, g(x,y) = 0 \quad \text{and} \quad q(x,y) = 0,$$

the point (x_1,y_1) is called a *singular point.*

Consider the equation $y^2 = x^3$. By implicit differentiation, $dy/dx = 3x^2/2y$. Hence, the origin is a singular point. At the origin the two branches $y = x^{3/2}$ and $y = -x^{3/2}$ meet, and both are tangent to the x-axis, since $dy/dx = \pm \frac{3}{2}\sqrt{x}$. In this case the singular point is called a *cusp.*

Consider the equation $y^2 = x^3 - 2x^2$. By implicit differentiation,

$$\frac{dy}{dx} = \frac{3x^2 - 4x}{2y}.$$

Here again, the origin is a singular point. By solving for y and differentiating explicitly, the slope for $x = 0$ is found to be imaginary. In this case the singular point is an *isolated* point.

To trace the curve corresponding to the equation

$$y^2 = x^2(4 - x^2),$$

we shall use the information obtained by applying both analytic and calculus methods.

The curve is symmetrical with respect to both coordinate axes. The x-intercepts are -2, 0 and 2. The y-intercept is 0. For the interval $-2 < x < 2$, y is real. For the intervals $x < -2$ and $x > 2$, y is imaginary.

There are two branches of the curve corresponding to the two functions

$$y = x\sqrt{4 - x^2}, \quad y = - x\sqrt{4 - x^2}.$$

Attending to the first branch,

$$\frac{dy}{dx} = \frac{4 - 2x^2}{\sqrt{4 - x^2}} \quad \text{and} \quad \frac{d^2y}{dx^2} = \frac{2x(x^2 - 6)}{(4 - x^2)^{3/2}}.$$

From these derivatives it is found that $(-\sqrt{2}, -2)$ is a minimum point, $(\sqrt{2}, 2)$ is a maximum point and $(0,0)$ is an inflection point. The curve intersects the x-axis perpendicularly at the points $(-2,0)$ and $(2,0)$. From this information the first branch of the curve is drawn in Figure 25 with the full line. An analysis of the second function yields derivatives differing from those obtained in sign only. With these results the curve is completed with the dotted line in the same figure. The singular point at the origin is a *double point*, since the slopes of the curve at this point are real and equal to 2 and -2.
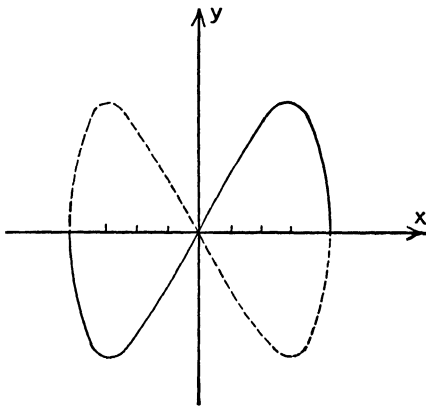

Fig. 25

The curve drawn in Figure 26 is obtained from the equation

$$x^2y - 2x + 4y = 0.$$

The derivatives, taken implicitly, are

$$\frac{dy}{dx} = \frac{2(1 - xy)}{x^2 + 4} \quad \text{and} \quad \frac{d^2y}{dx^2} = \frac{2(3x^2y - 4x - 4y)}{(x^2 + 4^2)}.$$

The curve has no singular point. The coordinates of the critical points are obtained by solving the given equation simultaneously with the equation $xy - 1 = 0$. They are found to be $(-2, -\frac{1}{2})$ and $(2, \frac{1}{2})$, of which the first

is a minimum point and the second is a maximum. The coordinates of the inflection points are obtained by solving the given equation simul-
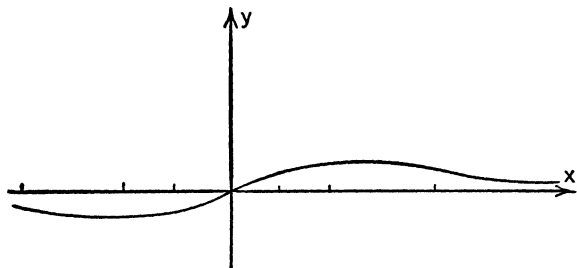

FIG. 26

taneously with the equation $3x^2y - 4x - 4y = 0$. They are found to be $(0,0)$ and $(\pm \sqrt{15}, \pm 2\sqrt{15}/19)$.

Exercise 23

GROUP A

Trace each of the following curves.

1. $xy + x + 1 = 0$.
2. $y^2(x + 1) = 1$.
3. $x \cdot y^2 + x^2 = 1$.
4. $y^2 = x(x^2 - 1)$.

5. $xy^2 + x = 4$.
6. $y^2(x^2 + 1) = 1$.
7. $x^2 - y^2 - 8x + 6y + 16 = 0$.
8. $y(x^2 + 1) = x^2 + 4$.

Find the equation of the tangent to each of the following curves at the specified points. Find the Cartesian equation of each.

9. $x = t^2$, $y = t - 1$ at $t = 2$.
10. $x = t^2 - 2$, $y = t^3 + 3$ at $t = 1$.
11. $x = 2t + 3t^2$, $y = 2t + 3$ at $t = -1$.

12. Find the values of t which give the critical points and the inflection point of the curve $x = 3t$, $y = t^3 - 3t^2$. Test the critical points for maxima and minima and trace the curve.

GROUP B

Trace each of the following curves.

13. $xy^2 - y^2 - 4x = 0$.
14. $x^2y + 4y - 2 = 0$.
15. $y(x - 1)^2 = 1$.
16. $y(x + 1) = (x - 2)^2$.
17. $y^2 = x^3 - 4x^2$.

18. $x^2y^2 = 4 - x^3$.
19. $y^2 = x(2 - x)^2(4 - x)$.
20. $y(x^2 - x - 2) = x$.
21. $y^2(a^2 - x^2) = ax$.
22. $y^2 = (x - 1)^3$.

In each of the following find the first and the second derivatives of y with respect to x as functions of the parameter.

23. $x = 3t^2$, $y = t^3$.
24. $x = t^2 + 2t$, $y = t^3$.

25. $x = \dfrac{1}{1 + t}$, $y = \dfrac{1}{1 - t}$.

26. Find the coordinates of the critical point of the curve $x = 3t^2$, $y = t^2 - 2t$. Find the range of values of t for which the curve is concave downward and concave upward. Trace the curve.

GROUP C

Trace each of the following curves.

27. $x^2y^2(1 - x) = a^2$.	**37.** $y^2(x^2 + 4) = a^2$.
28. $y^2(a^2 + x^2) = ax$.	**38.** $y(x^4 - a^4) = x$.
29. $y(x - a)^3 = ax^3$.	**39.** $x^4y(4 - x^2) = 1$.
30. $axy = (x - 2a)^3$.	**40.** $xy(x^2 - a^2) = b$.
31. $x^2y^3 = a^3$.	**41.** $y^2(x^2 - a^2)^3 = b$.
32. $x^4y^3 = a^4$.	**42.** $y(a^2 + x^2)^2 = bx$.
33. $y^2(1 - x^2) = a^2$.	**43.** $x = t^2 - 2$, $y = 4/t$.
34. $x^2y^2(x^2 - a^2) = b^2$.	**44.** $x = 4t^2$, $y = 4(1 - t)^2$.
35. $xy(x^2 + a^2) = b$.	**45.** $x^{2/3} + y^{2/3} = 4$.
36. $x^2y(x^2 + a^2) = b$.	

41. Motion in a Curve.

In Chapter II, Sections 18 and 21, the velocity and the acceleration on a particle moving on a straight line were studied. Such a motion is called *rectilinear motion*. In this section we are to consider *curvilinear motion*, or the motion of a particle along a curve.

Vectors. A *vector quantity* is defined as a quantity having a magnitude, a sense and a direction. Accordingly, such a quantity may be represented by a directed line segment where the length represents the magnitude of the vector quantity. The sense is indicated by an arrowhead at one extremity and the direction is the angle which the line segment makes with some line of reference. Such a line segment is called a *vector*. Vectors are of considerable importance in physics as they are used to represent forces, velocities, accelerations and other such quantities.

If two forces act simultaneously on the same particle, they are equivalent to a single force which is called their *resultant*. The two original forces are called the *components* of the resultant. If the two forces are represented by the vectors AB and AC as in Figure 27, the resultant force is represented by the vector AD which is the diagonal of the parallelogram $ABCD$. This is known as the *parallelogram law* of vectors. Conversely, a single vector can be resolved into two components in many ways. That is, in the same illustration, two sides of any
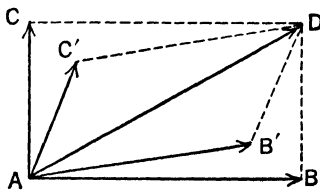

Fig. 27

parallelogram of which AD is the diagonal, as AB' and AC' in the figure, are two components of the same vector.

Velocity. If a moving particle describe a plane curve, its coordinates at any time may be expressed as functions of the time. Hence, the path of the particle has the parametric equations

$$x = f_1(t), \quad y = f_2(t)$$

in which the parameter t is the time elapsed. At any instant t, the particle has a position $P(x,y)$ on the curve, where x and y are the values corresponding to the value chosen for t.

The velocity of the particle at any instant in the *x-direction* is the rate of change of x with respect to t, or

$$v_x = \frac{dx}{dt} = f_1'(t).$$

And the velocity of the particle at the same instant in the *y-direction* is the rate of change of y with respect to t, or

$$v_y = \frac{dy}{dt} = f_2'(t).$$

If v represents the velocity of the particle along its path at any instant, v_x is called the *x-component of v* and v_y is called the *y-component of v*.
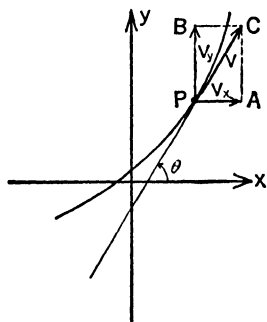

FIG. 28

Suppose that a particle is moving on the curve drawn in Figure 28 and that we wish to find the velocity along the path at the instant the particle reaches the point P. If the x- and the y-components of the velocity are represented by the vectors PA and PB, respectively, where v_x is drawn parallel to the x-axis and v_y is drawn parallel to the y-axis, the resultant velocity is represented by the vector PC which is the diagonal of the rectangle $PABC$. Let the direction of the vector PC be represented by the angle θ, that is,

$$\angle APC = \theta.$$

Then

$$v_x = v \cos \theta, \quad v_y = v \sin \theta.$$

Squaring these two equations and adding

$$v_x{}^2 + v_y{}^2 = v^2.$$

This result is a fundamental relation by means of which the magnitude of the velocity of the particle at any instant may be found from the components of that velocity.

Upon dividing the second equation by the first,

$$\frac{v_y}{v_x} = \frac{v \sin \theta}{v \cos \theta} = \tan \theta.$$

From Section 26,

$$\frac{dy}{dx} = \frac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}} = \tan \theta,$$

where $\tan \theta$ is the slope of the curve at the point P. Hence, the velocity of a particle moving on a curve at any instant t is represented by the vector v laid off on the tangent to the curve at the corresponding point $P(x,y)$.

The direction of the motion of a particle at any instant can be found from the components of the velocity at that instant by means of the relation

$$\theta = \arctan \frac{v_y}{v_x}.$$

Suppose that a particle moves so that its position at any time is given by the equations

$$x = t^2, \quad y = 3t - 1$$

and that we wish to find the velocity and the direction of the motion at the instant $t = 2$.

The solution is carried out as follows:

$$v_x = 2t, \quad v_y = 3.$$

$$|v| = \left. \sqrt{4t^2 + 9} \right]^{t=2} = 5.$$

$$\theta = \left. \arctan \frac{3}{2t} \right]^{t=2} = \arctan \tfrac{3}{4}.$$

The path of the motion is the parabola drawn in Figure 29. When $t = 2$, the particle has the position $P(4,5)$ and the vectors v_x, v_y and v have the lengths 4, 3 and 5, respectively.

Acceleration. We shall consider the acceleration of a moving particle which describes a plane curve in the same manner as has been followed for the study of its velocity. The acceleration of the particle at any instant

in the x-direction is the rate of change of v_x with respect to the time, or

$$j_x = \frac{dv_x}{dt} = f_1''(t).$$

And the acceleration at the same instant in the y-direction is the rate of change of v_y with respect to the time, or

$$j_y = \frac{dv_y}{dt} = f_2''(t).$$

If j represents the acceleration of the particle along its path at any instant, j_x is called the *x-component of j* and j_y is called the *y-component of j*.
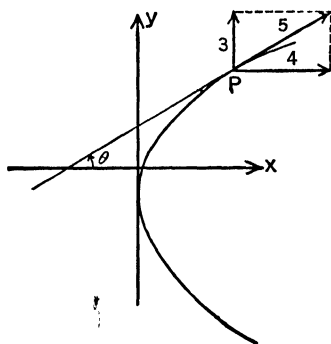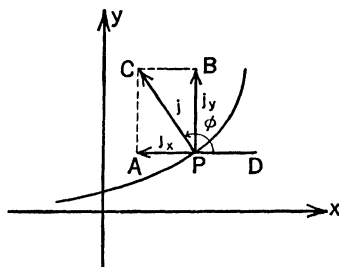

FIG. 29   FIG. 30

Suppose that a particle is moving on the curve drawn in Figure 30 and that we wish to find the acceleration along the path at the instant the particle reaches the point P. If the x- and the y-components of the acceleration are represented by the vectors PA and PB, respectively, where j_x is drawn parallel to the x-axis and j_y is drawn parallel to the y-axis, the resultant acceleration j is represented by the vector PC which is the diagonal of the rectangle $PABC$. Let the direction of j be represented by the angle ϕ, that is

$$\angle DPC = \phi.$$

Then $$j_x = j \cos \phi, \quad j_y = j \sin \phi.$$

Squaring these two equations and adding

$$j_x{}^2 + j_y{}^2 = j^2.$$

This result is the fundamental relation by means of which the magnitude

of the acceleration of the particle at any instant may be found from the components of that acceleration.

Upon dividing the second equation by the first,

$$\frac{j_y}{j_x} = \tan \phi.$$

Hence, the acceleration of a particle at any instant may be represented by a vector j laid off on a line having the direction given by the relation

$$\phi = \arctan \frac{j_y}{j_x}.$$

It is to be observed that it is possible to find the absolute values, only, of the magnitudes of the velocity and the acceleration from the relations

$$|v| = \sqrt{v_x{}^2 + v_y{}^2}$$

and

$$|j| = \sqrt{j_x{}^2 + j_y{}^2}.$$

The sense of the vectors representing them is determined from the directed vectors representing their components.

It is important here to note that, while the acceleration j of a particle having rectilinear motion is equal to dv/dt, this is not true for a particle having curvilinear motion. That is, the resultant acceleration is not represented by the derivative of the resultant velocity with respect to the time in curvilinear motion.

Exercise 24

GROUP A

A particle moves along a plane curve given by each of the following equations.

1. $x = 2t$, $y = 2t - 3t^2$. Find v_x, v_y, v and the coordinates of the position when $t = 1$. Draw the curve and the velocity vectors at the given time.

2. $x = 3t$, $y = 3t - 3t^2$. Find the position and the time at which the motion is parallel to the x-axis. Find the velocity and the direction of motion when $t = 2$.

3. $x = t^2$, $y = 2t - t^2$. Find j_x j_y, j and the coordinates of the position when $t = 2$. Draw the curve and the acceleration vectors at the given time.

4. $x = t^2$, $y = 2t^2 - 5t$. Find the velocity at any time. Find time and the position at which the velocity is minimum.

5. $x = 3t^2$, $y = 3t^2 - t^3$. Find the acceleration at any time. Find the time and the position at which the acceleration is minimum.

6. $x = t^2$, $y = (t - 2)^2$. Find v_x, v_y, v, j_x, j_y, j and the coordinates of the position when $t = 1$. Draw the velocity and acceleration vectors at the point found.

7. $x = t^3$, $y = t^2$. Find the directions of the velocity and the acceleration vectors. Find the Cartesian equation of the path.

8. $x = t^2 - 3$, $y = t^3 + 2$. Find the velocity, the position and the time at which the velocity is minimum. Find the Cartesian equation of the path and draw the vector representing the least velocity.

9. A man can row at the rate of 5 mi. per hr. and wishes to reach a point across a river $1\frac{1}{2}$ miles wide directly opposite his starting point. If the current of the river is 4 mi. per hr., in what direction must he row and how long will it take him?

10. A man can row 4 mi. per hr. If he rows across a river $1\frac{1}{2}$ miles wide at right angles to the current which is 3 mi. per hr., when and where will he reach the shore opposite?

GROUP B

Given the following equations of the paths of curvilinear motion. In each case find v_x, v_y, v, j_x, j_y, j, the direction of v and the direction of j at the given instant.

11. $x = 2t$, $y = t^3$ at $t = 2$.

12. $x = 3t^2$, $y = 2t^3$ at $t = 1$.

13. $x = 2t$, $y = \dfrac{1}{t^2}$ at $t = 1$.

14. $x = 1 + t$, $y = \dfrac{1}{1 - t}$ at $t = 2$.

15. $x = \dfrac{t}{1 + t}$, $y = \dfrac{1 - t}{t}$ at $t = 1$.

16. $x = at + b$, $y = ct + d$ at $t = t_1$.

17. $x = \dfrac{6t}{1 + t^3}$, $y = \dfrac{6t^2}{1 + t^3}$ at $t = 1$.

18. A particle describes the first quadrant of the circle $x^2 + y^2 = 20$ with $v_y = 3$, constant. Find v_x, v, j_x and j_y at the point $(2, 4)$.

19. A particle describes the upper half of the parabola $y^2 = 6x$, with $v = 5$. Find v_x, v_y, j_x and j_y at the point $(6, 6)$.

20. The equations of the path of a particle are $x = 4/t$, $y = t^2$. Find the time at which the velocity and the acceleration vectors are at right angles to each other.

42. Projectiles.

An object which is thrown in the air is called a *projectile*. The path which is traced during the flight of a projectile is approximately a portion of a parabola. Were there no forces acting except that of gravity, the path would be that of a true parabola. In our work we shall neglect such forces as air resistance and speak of the parabolic path of a projectile. Consequently, our results will be first approximations to those which might be obtained from the true path, or the *trajectory* of a projectile.

A projectile which is thrown horizontally from an elevation traces the arc of a parabola from the vertex to the ground, while one which is thrown upward at an acute angle with the horizontal, traces a portion of a parabola whose vertex is the highest point reached. In both cases the axis of the parabola is vertical.

In expressing analytically the path of a parabola, it is convenient to express the coordinates of the position of the body at any instant in terms

of the time elapsed as a parameter. If a projectile is thrown from the origin with an initial velocity v_0 and with an initial direction α with the x-axis, the parametric equations of its path are

$$x = v_0 t \cos \alpha, \quad y = v_0 t \sin \alpha - \tfrac{1}{2}gt^2.$$

From these equations the x- and the y-components of the velocity at any time are

$$v_x = v_0 \cos \alpha, \quad v_y = v_0 \sin \alpha - gt,$$

from which the magnitude of the velocity can be found. Again, from the last equations the x- and the y-components of the acceleration at any time are

$$j_x = 0, \quad j_y = -g,$$

from which it is seen that the acceleration is the constant g directed downward. We shall assume $g = 32$ ft. per sec.[2]

If the parameter is eliminated from the equations of the path of a projectile, the Cartesian equation is seen to be that of a parabola having the form

$$Ax^2 + Bx + Cy = 0.$$

This justifies the statements made in the first paragraph of this section.

The *range* of a projectile is the horizontal distance to the point where it strikes the ground.

Exercise 25

GROUP A

1. A projectile is given an initial velocity of 200 ft. per sec. in a direction inclined 45° with the horizontal. Write the equations of the path if it is released from the origin. Find the component and resultant velocities at the end of the third second.

2. A projectile is given an initial velocity of 150 ft per sec. in a direction inclined 60° with the ground. Find the height to which it rises, the range, the angle at which it strikes the ground and the velocity with which it strikes the ground.

3. Find the range and the time of falling of a projectile thrown horizontally from a height of 600 ft. with an initial velocity of 80 ft. per sec.

4. Show that the magnitude of the velocity of a projectile at any time is

$$\sqrt{v_0^2 - 2gv_0 t \sin \alpha + g^2 t^2}.$$

5. Show that a projectile reaches its maximum height when

$$t = \frac{v_0 \sin \alpha}{g}$$

and find the maximum height.

6. Show that the magnitude of the velocity has a minimum value of $v_0 \cos \alpha$ when the projectile has attained its maximum height.

7. Find the Cartesian equation of the path of a projectile.

8. Show that the range of a projectile thrown upward from the ground is twice the abscissa of the highest point in the path.

9. Show that the angle at which a projectile thrown from the ground strikes the ground is equal to the angle at which it was released.

10. Show that the magnitude of the velocity with which a projectile thrown upward from the ground strikes the ground is equal to the initial velocity.

11. Find the angle α at which a projectile should be released in order that the range be greatest for a given initial velocity

12. Derive the parametric equations of the path of a projectile using

$$j_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2} = 0, \quad j_y = \frac{dv_y}{dt} = \frac{d^2y}{dt^2} = -g,$$

and assuming the same initial conditions as heretofore.

GROUP B. Tangential and Normal Components of Acceleration.

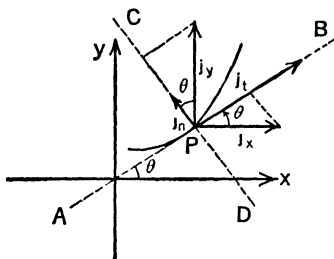Instead of resolving the acceleration of a particle moving along a plane

FIG. 31

curve into the components j_x and j_y, in the x- and the y-directions, respectively, it is often convenient to resolve it into components along the tangent and the normal line to the path. Referring to Figure 31, assume that j has the horizontal and vertical components j_x and j_y. Let θ be the inclination of the tangent to the curve at the point P in question and let the line CD be the normal to the tangent line AB also at the point P. For the tangential component of j, we have

(1) $$j_t = j_x \cos \theta + j_y \sin \theta.$$

And for the normal component of j, we have ˙

(2) $$j_n = j_y \cos \theta - j_x \sin \theta.$$

Since ˙

(3) $$v_x = v \cos \theta \quad \text{and} \quad v_y = v \sin \theta,$$

these components can be expressed as follows:

(4) $$j_t = \frac{v_x \dfrac{dv_x}{dt} + v_y \dfrac{dv_y}{dt}}{\sqrt{v_x{}^2 + v_y{}^2}},$$

$$(5) \qquad j_n = \frac{v_x \dfrac{dv_y}{dt} - v_y \dfrac{dv_x}{dt}}{\sqrt{v_x{}^2 + v_y{}^2}}.$$

13. Draw the Figure 31 and carry out the derivations for (1) and (2).

14. Draw the velocity vector parallelogram and derive the equations (3).

15. Derive equation (4) from (1).

16. Derive equation (5) from (2).

17. Show that by the differentiation of $\sqrt{v_x{}^2 + v_y{}^2}$ with respect to t, $j_t = dv/dt$.

18. Given $x = 3t + 1$, $y = 2t^2 - 1$. Find the tangential and normal components of the acceleration.

19. Given $x = 2 - 4t$, $y = 2t^2 - t$. Find the tangential and normal components of the acceleration when $t = 1$.

20. A particle moves along the curve $x = t^3$, $y = t^3 - 18t$. Find the point at which the tangential component of the acceleration is minimum.

43. Time Rates.

If a variable is expressed as a function of the time, its derivative with respect to the variable t is called the *time rate of change* of the function. We have previously expressed variable distance measurements as functions of time and have called their derivatives, velocities. In this section we wish to study the rates of change of other variable quantities with respect to time, of which the velocity of a moving body is a particular case.

Suppose that air is blown into a rubber balloon and that the balloon retains its spherical shape. Then the radius r and the volume V are variable, and *at all times* have the relation

$$V = \tfrac{4}{3}\pi r^3.$$

Each variable is a function of the independent variable t, even though such functions are not expressed. And for any value of t, each variable has a rate of change. The latter are

$$\frac{dV}{dt} \quad \text{and} \quad \frac{dr}{dt}.$$

The time rates of change of such variables are obtained by implicit differentiation as was done in Section 37. Hence,

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

If air is forced into a spherical balloon at the rate of 4π cu. ins. per sec., let us find the time rate of change of the radius when the radius is 10 ins.

It is given that

$$\frac{dV}{dt} = 4\pi \quad \text{and} \quad r = 10.$$

Hence, from the last equation we have

$$\frac{dr}{dt} = \frac{4\pi}{400\pi} = 0.01 \text{ in. per sec.}$$

It is observed that in this problem the time rate of change of the volume is constant, while the time rate of change of the radius is variable, differing for each value of r.
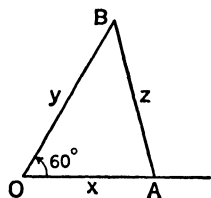
As a second illustration, two ships start simultaneously from the same position, one sailing due east at the uniform rate of 20 mi. per hr. and one sailing N 60° E at the uniform rate of 30 mi. per hr. How fast are they separating at the end of 2 hrs.?

In Figure 32 let $OA = x$ represent the distance the first ship has travelled after t hours, let $OB = y$ represent the distance the second ship has travelled during the same time and let $AB = z$ represent the distance between them. From the law of cosines

Fig. 32

$$z^2 = x^2 + y^2 - 2xy \cos 60°,$$
$$z^2 = x^2 + y^2 - xy.$$

Differentiating implicitly with respect to t,

$$2z\frac{dz}{dt} = (2x - y)\frac{dx}{dt} + (2y - x)\frac{dy}{dt}.$$

At the end of two hours, $x = 40$ and $y = 60$ from which $z = 20\sqrt{7}$ miles. It is given that

$$\frac{dx}{dt} = 20 \quad \text{and} \quad \frac{dy}{dt} = 30.$$

Hence, $$\frac{dz}{dt} = 10\sqrt{7} = 26.46 \text{ mi. per hr.}$$

As a final illustration, the solution in a conical funnel 16 ins. across the top and 10 ins. deep is dripping out at the rate of 4 cu. ins. per hour. How fast is the surface falling when the solution is 5 ins. deep?

The volume of the solution in the funnel at any time is

$$V = \frac{\pi}{3} x^2 y,$$

where x is the radius of the surface and y is the depth. From similar triangles $5x = 4y$. Eliminating x,

$$V = \tfrac{16}{75} \pi y^3.$$

Differentiating, $$\frac{dV}{dt} = \tfrac{16}{25} \pi y^2 \frac{dy}{dt}.$$

Hence, $$\frac{dy}{dt} = \frac{1}{4\pi} \text{ in. per hr.}$$

Exercise 26

GROUP A

1. Water is running into a vertical cylindrical tank having a radius of 10 ft. at the rate of 4 cu. ft. per min. How fast is the surface rising?

2. A man is walking toward a vertical flagstaff at the rate of 6 ft. per sec. When he is 60 ft from the foot, how fast is he approaching the top of the flagstaff?

3. Water flows at the rate of 2 cu. ft. per min. into a vessel in the form of an inverted right circular cone of altitude 6 ft. and radius of the base 2 ft At what rate is the surface rising when the surface is halfway to the top?

4. A circular metal plate is heated and expands retaining its circular shape. If the radius increases 0 01 in per min., how fast are the area and the circumference of the plate increasing when the radius is 10 ins ?

5. Two automobiles start simultaneously at the same point, one travelling due west at the rate of 30 mi. per hr , and one due south at the rate of 40 mi. per hr. How fast are they separating at the end of 3 hrs ?

6. A particle is moving along the parabola $y^2 = 4x$ so that when $x = 4$ the abscissa is increasing 6 ins per min. At what rate is the ordinate increasing and what is the velocity of the particle in its path?

7. The top of a ladder 30 ft. long slides down a vertical wall at the rate of 4 ft. per sec. How fast does the foot slide away from the wall on a horizontal floor when the foot is 20 ft. from the wall?

8. A cubical block of ice is melting, retaining its cubical shape. How fast is the surface decreasing, if the volume decreases at the rate of 2 cu. ins. per min. when the length of an edge is 20 ins.?

9. A man 6 ft. tall is walking at the rate of 8 ft per sec. directly away from a lamp post 20 ft. high. How fast is the tip of his shadow moving and how fast is his shadow lengthening?

10. A particle moves on the circle $x^2 + y^2 = 100$ so that x decreases at the rate of 12 ins. per sec. when $x = 6$. Find the rate of change of the ordinate.

GROUP B

11. A trough has the form of a right prism whose vertical ends are equilateral triangles. If the length of the trough is 15 ft., and if the water leaks out at the rate of 2 cu. ft. per hr., how fast is the surface falling when the water is 6 ins. deep?

12. Three towns A, B and C are connected by straight roads. The road AC is 20 miles long and makes an angle of 30° with the road AB. One automobile starts at C travelling toward A at the rate of 15 mi. per hr. and at the same time a second automobile starts at A travelling toward B at the rate of 24 mi. per hr. How fast are the two approaching each other 20 minutes later?

13. A balance wheel 6 ft. in diameter is rotating at the rate of 50 revolutions per minute. A lug on the rim of the wheel becomes loosened and flies off at a tangent. Assuming that no allowance is made for any new force, how fast will the lug be moving from the center of the wheel 3 seconds later?

14. The top of a ladder 30 ft. long rests against the vertical side of a house and the foot is being pulled from the wall at the rate of 5 ft. per sec. along a roof which makes an angle of 120° with the vertical wall. How fast is the top of the ladder sliding down the wall?

15. Coal is being released from a hopper at the rate of 12 cu. ft. per min. and forms a right circular conical pile whose height is always half the diameter of the base. How fast is the ground being covered with coal when the radius of the base is 20 ft.?

16. A light on the floor is 50 ft. from a wall. A man 6 ft. tall walks from the light toward the wall at the rate of 4 ft. per sec. Find the rate at which the length of his shadow on the wall is changing when he is 10 ft. from the wall.

17. A right triangle ABC has the right angle at C and $AC = 50$ ins., $CB = 40$ ins. A particle starts at A and moves toward C at the rate of 20 ins. per sec. At the same time a second particle starts at B and moves in a line parallel to AC and in the opposite direction at the rate of 30 ins. per sec. At what rate are they approaching or separating at the end of $\frac{1}{2}$ second and at the end of 2 seconds? At what time are they nearest each other?

18. Points A and B slide along a straight line on a horizontal plane and are connected by a cord 84 ins. long passing over a pulley at C which is 30 ins. above the line AB. If B is pulled away from the point D in AB vertically below C at the rate of 20 ins. per sec., how fast is A moving toward D when $DB = 40$ ins.

19. An airplane leaves the ground at an angle of 30° at a distance of 2 miles from an observer. If the speed of the airplane is 70 mi. per hr., at what rate is it approaching or receding from the observer according as it flies directly toward or away from him?

20. An elevated track is 36 ft. above a street and crosses it at right angles. A car on the track travelling at the rate of 27 ft. per sec. reaches the crossing at the same instant that an automobile travelling at the rate of 24 ft. per sec. reaches it. How fast are they separating after one second?

44. Reversal of Time Rates.

A study of the integration of polynomials was presented in Chapter IV. It was also stated that

$$\int t^n \, dt = \frac{t^{n+1}}{n+1} + C,$$

which can be applied for all real values of n except -1. In this section it

is desired to consider some applications which are the reverse of those treated in the last sections.

As a first illustration of the nature of such problems, suppose that water is being pumped into a pressure tank so that the volume is increasing at the rate of $8/\sqrt{t}$ cu. ft. per minute. We wish to know the volume of water pumped into the tank during the first four minutes and during the second four minutes.

It is given that

$$\frac{dV}{dt} = \frac{8}{\sqrt{t}}.$$

From the definition of the differential of a function,

$$dV = \frac{8}{\sqrt{t}}\, dt.$$

Hence,
$$V = 8\int t^{-1/2}\, dt = 16\sqrt{t} + C.$$

From $t = 0$ to $t = 4$, $\quad V = 16(2 - 0) = 32$ cu. ft.

From $t = 4$ to $t = 8$, $\quad V = 16(2\sqrt{2} - 2) = 13.25$ cu. ft.

As a second illustration, a particle describes a plane curve in such a way that the x- and the y-components of the acceleration are 2 and $6t$ ft. per sec. per sec., respectively. If the particle have the position $(2,0)$ when $t = 0$ and, $(10,6)$ when $t = 2$ secs., we may find the parametric equations of the path.

It is given that

$$j_x = \frac{dv_x}{dt} = 2, \quad j_y = \frac{dv_y}{dt} = 6t.$$

Integrating each equation,

$$v_x = 2\int dt = 2t + A_1$$

$$v_y = 6\int t\, dt = 3t^2 + B_1.$$

Again, integrating each equation,

$$x = \int (2t + A_1)dt = t^2 + A_1 t + A_2$$

$$y = \int (3t^2 + B_1)dt = t^3 + B_1 t + B_2.$$

For $t = 0$, $x = 2$ and $y = 0$ gives $A_2 = 2$ and $B_2 = 0$.

For $t = 2$, $x = 10$ and $y = 6$ gives $A_1 = 2$ and $B_1 = -1$.

Hence, the required parametric equations of the path are

$$x = t^2 + 2t + 2, \quad y = t^3 - t.$$

Exercise 27

GROUP A

1. Water is flowing into a reservoir at the rate of 200 cu. ft. per min. How much flows in during the first 10 mins.? How much flows in during any 10 minute period?

2. A mountain stream empties into a lake. Melting snow increases the volume of the flow at the rate of $60t$ cu. ft. per hr. Find the increased volume of water in the lake during the first 10 hours and during the second 10 hours

3. An automobile is travelling at the rate of 30 mi per hr. If it is accelerated 10 mi. per hr. per hr., how far will it travel during the first 2 hours and during the second 2 hours?

4. The x- and the y-components of the velocity of a moving particle are 4 and $2t$, respectively. If the particle starts motion at the point $(3,2)$, find the parametric equations of the path.

5. The x- and the y-components of the acceleration of a moving particle are 4 and 0, respectively. If the particle starts motion at the origin with the x- and the y-components of the velocity 3 and 6, respectively, find the parametric equations of the path.

6. A particle starts motion at the point $(3,-2)$ with $v_x = 5$ and $v_y = -4$. If $j_x = 2t$ and $j_y = 3$, find the parametric equations of the path.

7. A particle starts motion at the point $(0,2)$ and it is given that $x = t^2$ and $v_y = 3t^2$. Find the Cartesian equation of the path and the acceleration in the path when $t = 1$.

8. A particle moves along a straight line with $j = t$. It is at rest at $x = 1$ when $t = 1$. Find the equation for its motion and its initial position and velocity. Find its position and velocity when $t = 3$.

9. A car is moving northward at the rate of 40 mi. per hr. If a ball is thrown from the car toward the east at 20 ft. per sec., find the direction and velocity of the ball with reference to the ground.

10. Find the magnitude and direction of the resultant of a force of 15 units directed northeast and a force of 8 units directed $30°$ W of N.

GROUP B

11. A particle moves on a straight line with $j = t - 1$ and starts from the origin with a speed of 1 ft. per sec. toward the left. Find the equation of motion, when it will stop and when it will return to the origin.

12. A particle moves so that $x = t^2$ and $y = t/2$. Find the Cartesian equation of its path, its velocity and its acceleration.

13. A point moves along the curve $xy = a$ with the vertical component of the velocity constant. Find the acceleration.

14. A particle starts motion at the origin, moving along the curve $2y = x^2$. As it passes through the point $(2,2)$, the x-component of the velocity is 3. Find the resultant velocity and show that $2j_x - j_y + 9 = 0$.

15. Given $j_x = k$ and $j_y = 0$. If the particle starts at the origin with $v_x = a$ and $v_y = b$, find the equations of the path.

16. Given $j_x = a$ and $j_y = b$. If the particle starts at the origin with $v = c$ directed along the y-axis, find the equations of the path.

17. Given $j_x = 1/v_x$ and $j_y = 0$ If the particle starts motion at the point $(9,9)$ with $v_x = 3$ and $v_y = 2$, find the equations of the path.

18. A car starts from rest with an acceleration of 16 rds. per min. per min. for 1 minute. Then it coasts for 2 minutes.   Thereafter the brake is applied giving a retardation of 16 rds per min per min., bringing the car to a stop.   Find the distance covered by the car and the time required.

19. An automobile is moving at the rate of 8 rds. per min. and is given an acceleration of 16 rds. per min. per min for 3 minutes.   Then it coasts for 3 minutes.   Thereafter, the brake is applied giving a retardation of 28/3 rds. per min. per min. bringing it to a stop   Find the distance covered and the time required.

GROUP C

20. Find the area bounded by $y = x^2$, $y = 2x$ and $y = x$.

21. A road is to be built from a point A, located on the corner of a large rectangular tract of timber, to a point B situated in the timber 50 rds south and 30 rds. east of A.   The cost of building the road along the edge of the timber is one-half as much per unit of length as through the timber.   Find the length of the road for which the cost of building is minimum.

22. Prove that for a given hypotenuse, the isosceles right triangle has the maximum area.

23. A point is moving on the curve $x^2 + 4y^2 = 16$.   Find the coordinates of the points at which x is increasing at the same rate y is decreasing.

24. A point is moving in the counterclockwise direction around the circle $x^2 + y^2 = 100$ with a constant speed of 6 ins. per sec.   Find the components of its velocity at the point $(6,8)$ and at $(8, -6)$.

25. One ship is 75 miles due east of a second ship   The first is sailing due west at the rate of 9 miles per hour and the second, due south at the rate of 12 miles per hour. Find how long they continue to approach each other and find the nearest distance they can get to each other.

26. A particle moves downward on the right-hand branch of the curve $xy = 12$ with a constant total velocity of 5.   Find the components of the velocity and the acceleration at the point $(4,3)$ and draw the vector of the resultant acceleration at that point showing magnitude and direction.

27. Approximate the value of $(x - 1)^3 + \dfrac{2}{(x - 2)^2} - \sqrt{x + 1}$ for $x = 3.02$, using differentials.

CHAPTER VI

45. The Definite Integral.

In Chapter IV an indefinite integral of a polynomial was defined as a function whose derivative is the given polynomial.   Thus

$$\int f(x)\ dx\ =\ F(x)\ +\ C,$$

where $f(x)$ is called the *integrand*.   In this chapter, also, the integrand is limited, with a few exceptions, to polynomials.   The integration of a more general class of functions is given consideration in Chapter IX.   Consequently, in the discussions which follow, the integrand $f(x)$ is assumed to be a continuous single-valued function.

An indefinite integral of a function does not have a definite value corresponding to each value of its variable because of the constant of integration C, hence its name, *indefinite integral*.

If the symbols V_a and V_b are chosen to represent indefinite integrals of $f(x)dx$ for $x = a$ and $x = b$, respectively, one is a function of a and C and the other is a function of b and C.   That is,

$$V_a = \int f(x)\ dx \Big]^{x=a} = F(x) + C \Big]^{x=a} = F(a) + C$$

and
$$V_b = \int f(x)\ dx \Big]^{x=b} = F(x) + C \Big]^{x=b} = F(b) + C.$$

From these two expressions it is possible to find a value of the difference which is independent of the constant C.   Thus,

$$V_b - V_a = F(b) - F(a).$$

This difference is a function of the constants a and b which generally exists, provided that $f(x)$ is a continuous single-valued function in the interval $a \leqq x \leqq b$.

The symbol for the *definite integral* is

$$\int_a^b f(x)\ dx,$$

which is read "the definite integral of $f(x)\,dx$ from a to b." It is *the difference between values of indefinite integrals for* $x = b$ *and* $x = a$. Hence, from this definition,

$$\int_a^b f(x)\,dx = F(x)\Big]_{x=a}^{x=b} = F(b) - F(a).$$

Thus, for a given continuous single-valued function the definite integral is a constant which depends only on a and b, hence its name, definite integral.

In the use of the definite integral, care should be exercised in placing the *limits* a and b. For the integral above, the limits are *from a to b*, while in the integral

$$\int_b^a f(x)\,dx = F(a) - F(b),$$

the limits are *from b to a*.

An interchange of the limits of a definite integral, changes the sign of the value of that integral. Thus,

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$$

Exercise 28

GROUP A

1. If $dy/dx = 2x$, show that $dy = 2x\,dx$ and express y as a function of x. Find the change in y from $x = 2$ to $x = 4$. Show that this change is equal to Δy.

2. If $dy = (3x^2 - 8x)\,dx$, express the change of y between the limits $x = 0$ and $x = 2$ as a definite integral and find its value.

Evaluate each of the following integrals.

3. $\displaystyle\int_0^2 4x\,dx.$

4. $\displaystyle\int_{-2}^2 3x^2\,dx.$

5. $\displaystyle\int_0^{16} \sqrt{x}\,dx.$

6. $\displaystyle\int_0^8 x^{2/3}\,dx.$

7. $\displaystyle\int_3^6 (t+1)\,dt.$

8. $\displaystyle\int_1^2 \frac{1+t^2}{t^2}\,dt.$

9. $\displaystyle\int_1^3 (1+y+y^2)\,dy.$

10. $\displaystyle\int_{10}^{100} (0.01 + 0.02\theta)\,d\theta.$

GROUP B

11. A stone falls vertically with a velocity given by $v = 12 + 16t$ ft. per sec. Find the distance covered as a function of t. Find the distance covered from $t = 2$ to $t = 5$ secs.

12. An automobile travels at the rate of $t/2$ ft. per sec. Find the distance covered during the first 10 secs and during the second minute.

13. If the pressure on a certain gas is increasing at the rate of $\sqrt{t}$ lbs. per sec., find the change in pressure from $t = 1$ to $t = 4$ secs.

14. A rocket is let fall from a parachute with an acceleration of 32 ft. per sec. per sec. Find the change in the velocity from $t = 2$ to $t = 3$ secs. Find the distance covered during the same time.

15. Find the area between the curve $y = x^2(1 - x)$ and the x-axis.

16. Find the area under the curve $y^3 = x^2$ from $x = -a$ to $x = a$.

17. Find the rate of change of the slope of the curve $y = x^3$ with respect to x. Express the change of the slope as a definite integral between the limits $x = 2$ and $x = 3$ and find that change.

18. Express the area in the first quadrant under the curve $x = 2t$, $y = t^2$ from $x = 0$ to $x = 4$ as a definite integral of $f(t)\,dt$ with values of t as limits and evaluate the integral.

19. For a continuous curve $x = f(y)$ in the first quadrant prove that $dS/dy = x$, where S is the area between the curve and the y-axis. From this result show that the area between the curve and the y-axis from $y = a$ to $y = b$ is

$$\int_a^b f(y)\,dy.$$

20. Find the area between the curve $y^2 + 4x - 4y = 0$ and the y-axis

46. Area under a Curve.

It was proved in Section 31 that if $f(x)$ is a continuous single-valued function, the rate of change of the area S between the curve $y = f(x)$ and the x-axis is given by

$$\frac{dS}{dx} = f(x).$$

From this relation we may now use the definite integral to find the area between a curve and the x-axis, provided that the right and left boundaries are known.

If $f(x)$ is positive in the interval $a \leq x \leq b$, the area S between the curve $y = f(x)$ and the x-axis from $x = a$ to $x = b$ is

$$S = \int_a^b f(x)\,dx = F(b) - F(a).$$

If $f(x)$ is negative in the same interval, the area S, is

$$S = -\int_a^b f(x)\,dx.$$

However, if the curve intersects the x-axis once at $x = c$ between $x = a$ and $x = b$ and if $f(a)$ is positive and $f(b)$ is negative, the area S, is

$$S = \int_a^c f(x)\,dx - \int_c^b f(x)\,dx.$$

When working with areas for curves whose equations are expressed parametrically, it is unnecessary to find the corresponding rectangular equations of those curves. If x and y are functions of the parameter t, the integrand and the differential may be expressed as functions of t, if so, the *limits must be expressed as t limits.*

Consider the area in the first quadrant bounded by the curve

$$x = t^2, \quad y = t^3,$$

the x-axis and the ordinates $x = 1$ and $x = 4$.

From the first given equation $dx = 2t\,dt$. For $x = 1$, $t = \pm 1$ and for $x = 4$, $t = \pm 2$. Choosing the positive values of t, since the first quadrant area is desired, we have

$$S = \int_1^4 y\,dx = \int_1^2 2t^4\,dt,$$

or

$$S = \frac{2}{5} t^5 \Big]_{t=1}^{t=2} = \frac{62}{5} \text{ sq. units.}$$

Exercise 29

GROUP A

Find the area bounded by each of the following curves, the x-axis and the given ordinates.

1. $2y = x^2$ from $x = 1$ to $x = 4$.
2. $y = 6x - x^2$ from $x = 0$ to $x = 2$.
3. $y^2 = x^3$ in the first quadrant from $x = 0$ to $x = 9$.
4. $y = x^3 - 4x^2 + 3x$ from $x = 0$ to $x = 3$.
5. $y = \dfrac{x^3 + 4}{x^2}$ from $x = 2$ to $x = 4$.
6. $x = t + 1$, $y = t - 1$ from $x = 1$ to $x = 6$.
7. $x = \dfrac{t-1}{t}$, $y = \dfrac{t^3}{8}$ from $t = 2$ to $t = 4$.
8. $x = 1 + \sqrt{t}$, $y = 2t^2$ from $t = 0$ to $t = 9$.
9. $x = 1 - t$, $y = \sqrt{1+t}$ from $x = -2$ to $x = 2$.
10. $x = \sqrt{t+1}$, $y = \sqrt{t^2 - 1}$ from $t = 1$ to $t = 5$.

GROUP B

Find the area in each of the following problems.

11. Bounded by $y = 9 - x^2$ and the x-axis.
12. Bounded by $y = 4x - x^3$ and the x-axis.
13. Bounded by $y = 4x^2 - 9x^2 + 6x$, its minimum ordinate and the x-axis.
14. Bounded by $\sqrt{x} + \sqrt{y} = \sqrt{2}$ and the axes.
15. Bounded by $y^2 = 4x$ and $3y = x$.

16. Bounded by $y = x^2 - 4x$ and $y - x = 0$.

17. Bounded by $x = 2t$, $y = 8t^2$ and the line $y = 8$.

18. A curve cuts any line joining any point on it to the origin at right angles. Find the equation of the curve if it passes through the point $(4, -3)$.

19. Find the equation of the tangent to $x^{2/3} + y^{2/3} = 2a^{2/3}$ at the point (a, a).

20. An isosceles triangle is inscribed in the ellipse $b^2x^2 + a^2y^2 = a^2b^2$, where $a > b$, having its vertex at an extremity of the minor axis. Find the base and the altitude of the maximum triangle.

47. Summation Notation.

In mathematics the Greek letter $\sum$ is used to indicate the sum of any number of terms. For example,

$$\sum_{i=1}^{i=4} i = 1 + 2 + 3 + 4.$$

The letter i is always assumed to take on the values of the consecutive positive integers. In the sum the first value of i is indicated at the base of the symbol and the last value of i, at the top of the symbol.

The letter i may appear in various positions in the expression which is to be summed, that is, it may represent a term, a subscript or an exponent. In illustration, the following sums are written:

$$\sum_{i=2}^{i=5} (2 + i) = (2 + 2) + (2 + 3) + (2 + 4) + (2 + 5).$$

$$\sum_{i=1}^{i=n} x_i = x_1 + x_2 + x_3 + \cdots + x_{n-1} + x_n.$$

$$\sum_{i=0}^{i=3} x^i = 1 + x + x^2 + x^3.$$

$$\sum_{i=3}^{i=6} f(i) = f(3) + f(4) + f(5) + f(6).$$

$$\sum_{i=1}^{i=3} f(x_i) = f(x_1) + f(x_2) + f(x_3).$$

If $x_1 = 1, x_{10} = 19$ and $x_2 - x_1 = x_3 - x_2 = \cdots = x_{10} - x_9$, then

$$x_i - x_{i-1} = 2$$

and

$$\sum_{i=1}^{i=10} x_i = 1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 + 19.$$

Again, if $x_1 = 2, x_n = 12, n = 6, f(x) = x^2 + 2$ and $x_2 - x_1 = x_3 - x_2 = \cdots = x_n - x_{n-1}$, then $x_i - x_{i-1} = 2$ and

$$\sum_{i=1}^{i=n} f(x_i) = 6 + 18 + 38 + 66 + 102 + 146.$$

Exercise 30

Find the numerical value of each of the following sums.   In each case it is assumed that $x_2 - x_1 = x_3 - x_2 = x_n - x_{n-1}$.

1. $\displaystyle\sum_{i=1}^{i=6} 2x_i$, where $x_1 = 5$ and $x_6 = 10$.

2. $\displaystyle\sum_{i=1}^{i=5} x_i{}^2$, where $x_1 = 1$ and $x_5 = 5$.

3. $\displaystyle\sum_{i=1}^{i=6} f(x_i)$, where $f(x) = x - 3$, $x_2 = 3$ and $x_6 = 15$.

4. $\displaystyle\sum_{i=1}^{i=9} f(x_i)$, where $f(x) = x + 3$, $x_1 = 4$ and $x_{10} = 13$.

5. $\displaystyle\sum_{i=1}^{i=4} f(x_i)$, where $f(x) = 1/x$, $x_1 = 1$ and $x_4 = 4$.

6. $\displaystyle\sum_{i=1}^{i=10} \frac{1}{1 + x_i{}^2}$, where $x_1 = 0$ and $x_{11} = 1$.

7. $\displaystyle\sum_{i=1}^{i=10} \sin x_i$, where $x_1 = 0^r$ and $x_{11} = \pi^r/2$, and where x is measured in radians.

8. $\displaystyle\sum_{i=1}^{i=10} \cos x_i$, where $x_1 = 0^r$ and $x_{11} = \pi^r/2$, and where x is measured in radians.

9. $\displaystyle\sum_{i=1}^{i=5} \log x_i$, where $x_1 = 2$ and $x_5 = 10$.

10. $\displaystyle\sum_{i=1}^{i=n} y_i$, where $y = x^3 - x + 6$, $y_1 = f(0)$, $y_3 = f(6)$ and $n = 3$.

48. Approximate Area under a Curve by Summation.

Consider a function $y = f(x)$ which is positive, continuous and single-valued in the interval from $x = a$ to $x = b$ and draw the curve as in Figure 33.   Let S represent the area under the curve between the ordinate AC drawn at $x = a$ and the ordinate BP drawn at $x = b$.   It is desired to find an expression for an *approximation* to the area S.   This is done in one way by dividing the line segment AB into n equal intervals Δx, where

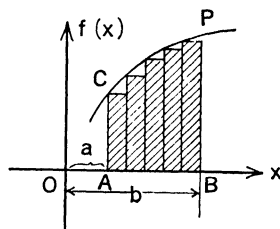$$\frac{b - a}{n} = \Delta x,$$

Fig. 33

and by erecting ordinates at each of the points of division.   At each point on the curve so located, parallels are drawn to the x-axis, thus forming a

series of *inscribed* rectangles. The sum of the areas of these inscribed rectangles is an approximation to the area S. The larger the value of n is chosen, or the smaller the value of Δx, the better is the approximation. In the figure n was chosen as 5, but were $n = 50$, $n = 100$, $n = 200$, etc., the approximations would be nearer and nearer to the value of S.

Let S_1 represent an approximation to the area S by the sum of n inscribed rectangles. Then

$$S_1 = \sum_{i=1}^{i=n} f(x_i)\, \Delta x = \left[f(x_1) + f(x_2) + f(x_3) + \cdots + f(x_n) \right] \Delta x,$$

where $\qquad x_1 = a, \quad x_2 = a + \Delta x, \quad x_3 = a + 2\,\Delta x, \cdots$

and $\qquad x_n = a + (n-1)\,\Delta x = b - \Delta x.$

Another way in which an approximation to the area S may be expressed is by using the *circumscribed* rectangles, as shown in Figure 34. Let S_2 represent such an approximation. Then

$$S_2 = \sum_{i=2}^{i=n+1} f(x_i)\, \Delta x = \left[f(x_2) + f(x_3) + \cdots + f(x_{n+1}) \right] \Delta x,$$

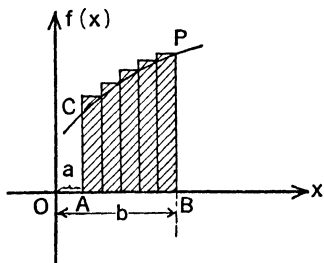where a, b, n and x_i are subjected to the same conditions as above.

FIG. 34

The first approximation S_1 is smaller than S and the second approximation S_2 is larger than S, no matter how large n is chosen, if, as we have assumed, $f(x)$ is an increasing function. These differences are due to the triangular shaped areas which are neglected in the first case and those which are added in the second case. Hence, for any value of n,

$$S_1 < S < S_2.$$

The inscribed or the circumscribed rectangles

$$f(x_i)\, \Delta x$$

are called *elements* of the area S. As n increases indefinitely, each of the elements approaches zero, that is, they are infinitesimals.

In the next section the limit of the sum of the elements of the area S as Δx approaches zero is considered, where it is shown that the $\lim\limits_{\Delta x \to 0} S_1$ or $\lim\limits_{\Delta x \to 0} S_2$ is equal to the area S. The problem of finding the limit of the sum of infinitesimal elements arises many times in the applications of the integral

calculus. Consequently, it is well that this first appearance of the subject should be treated with some care.

Exercise 31

1. Draw the curve $y = 1/x^2$ to a large scale and draw both the inscribed and the circumscribed rectangles for the area under the curve from $x = 0.5$ to $x = 1$ by taking $n = 5$. Compute the approximate area under the curve between the given limits. Compare the results with that obtained by the use of the definite integral.

2. Compute the approximate area under the curve in Problem 1 between the same limits by taking $n = 10$. Compare the results obtained with those found in Problem 1.

3. Draw the curve $y = 1/(x + 1)$ to a large scale and approximate the area under the curve from $x = 0$ to $x = 1$ by taking $n = 10$ and by using both inscribed and circumscribed rectangles.

In each of the following problems proceed as in Problem 2.

4. $y = 1/x$ between $x = 0.5$ and $x = 1$, taking $n = 10$.
5. $y = 1/(x^2 + 1)$ between $x = 0$ and $x = 1$, taking $n = 10$.
6. $y = \cos x$ for the first half arch, taking $n = 10$.
7. $y = \sin x$ for one arch, taking $n = 20$.
8. $y = \log x$ between $x = 1$ and $x = 2$, taking $n = 10$

49. Area under a Curve as the Limit of a Sum.

In Figure 35 let S_1 represent the sum of the inscribed rectangles and S_2 represent the sum of the circumscribed rectangles to the area S under the curve between the ordinates AC and BP. As in the previous section, it is assumed that there are n rectangles in each case. Then it is obvious that

$$S_1 < S < S_2$$

for all values of n. The difference between the area S_2 and the area S_1 is the sum of the small rectangles which in the figure, are the shaded ones. The sum of these rectangles is equivalent to the area of the rectangle $DEPQ$, as may be seen by sliding each small rectangle across into the last column. Since,
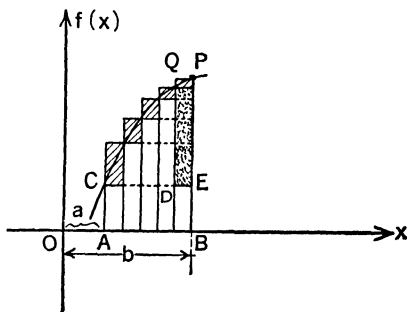

Fig. 35

$$EP = f(b) - f(a),$$

$$S_2 - S_1 = \left[f(b) - f(a) \right] \Delta x.$$

As n increases without limit, Δx approaches zero, so that

$$\lim_{\Delta x \to 0} (S_2 - S_1) = 0.$$

But since S always lies between S_1 and S_2, S_1 and S_2 approach S as their common limit.   Hence,

$$\lim_{\Delta x \to 0} \sum_{i=1}^{i=n} f(x_i) \,\Delta x = S.$$

The area under a curve between given limits is equal to *the limit of the sum of elementary rectangles inscribed or circumscribed to that area, as the number of such rectangles is increased without limit.*

In the foregoing discussion it was assumed that $f(x)$ was positive in the interval from $x = a$ to $x = b$ and was increasing with x so that the curve $y = f(x)$ rises from left to right.   In the following sections it is shown that these restrictions may be removed, that $f(x)$ may decrease as x increases and that $f(x)$ may alternately increase and decrease.   However, the assumption that $f(x)$ be a continuous single-valued function through the range, of course, cannot be abandoned.

50. The Fundamental Theorem.

Let us again assume that $f(x)$ is a continuous single-valued function and that it is a positive increasing function in the range $a \leqq x \leqq b$.   If S is the area bounded by the curve $y = f(x)$, the x-axis and the ordinates $x = a$ and $x = b$, it has been proved that

$$S = \int_a^b f(x) \, dx$$

and that

$$S = \lim_{\Delta x \to 0} \sum_{i=1}^{i=n} f(x_i) \,\Delta x.$$

Therefore, for the same area,

$$\lim_{\Delta x \to 0} \sum_{i=1}^{i=n} f(x_i) \,\Delta x = \int_a^b f(x) \, dx,$$

where $x_1 = a$, $x_n = b - \Delta x$ and $\Delta x = (b - a)/n$.   This equality, under the conditions imposed, is a statement of the *fundamental theorem of the integral calculus.*

While the fundamental theorem has been proved by means of a plane area, it is a general theorem, which is stated presently.   The general proof

of this theorem is beyond the scope of this treatment, such a proof may be found in an advanced calculus textbook.

Consider the area under the line $y = 2x + 2$ from $x = 0$ to $x = 2$. See Figure 36.

The interval OA from $x = 0$ to $x = 2$ is divided into n equal parts Δx. At each point of division vertical lines are drawn, thus dividing the area S into n increments ΔS. Then the area S is equal to

$$\sum_{i=1}^{i=n} \Delta S_i,$$

regardless of the value of n. Hence,

$$S = \lim_{n \to \infty} \sum_{i=1}^{i=n} \Delta S_i$$

Fɪɢ. 36

From the figure the area of any element is

$$\Delta S_i = y_i \, \Delta x + \tfrac{1}{2} \Delta y \, \Delta x = y_i \, \Delta x + \overline{\Delta x}^2$$

since $\Delta y = 2 \, \Delta x$ from the equation of the line. Hence,

$$S = \lim_{\Delta x \to 0} \sum_{i=1}^{i=n} (y_i \, \Delta x + \overline{\Delta x}^2).$$

If we disregard the infinitesimal of the second order, $\overline{\Delta x}^2$, and use only the principal part, $dS_i = y_i \, \Delta x$, of the infinitesimal of ΔS_i we shall have the exact area S. Thus,

$$S = \lim_{\Delta x \to 0} \sum_{i=1}^{i=n} y_i \, \Delta x.$$

By the fundamental theorem,

$$S = \int_0^2 y \, dx = \int_0^2 (2x + 2) \, dx = 8.$$

In this illustration, the step in which we substituted the differential of the area for the increment is justifiable. However, its validity depends on a theorem called Duhamel's Theorem, the proof of which may be found in any treatment of the advanced calculus.

The fundamental theorem of the integral calculus may be stated as follows: If an arbitrarily close approximation to a quantity R is given by
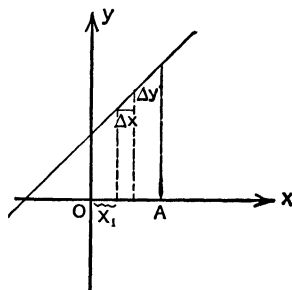
$$\sum_{i=1}^{i=n} f(x_i) \, \Delta x,$$

by taking n sufficiently large, so that

$$R = \lim_{n \to \infty} \sum_{i=1}^{i=n} f(x_i)\,\Delta x,$$

then

$$R = \int_a^b f(x)\,dx,$$

where $x_1 = a$, $x_n = b - \Delta x$ and $\Delta x = (b - a)/n$, provided that $f(x)$ is a function which is integrable.

In the applications of the fundamental theorem in which R is to be determined, it is convenient to divide it into n increments and to denote the approximation of each of the increments ΔR by its principal infinitesimal dR. It is assumed in each case that such an element can be expressed in the form $f(x)\,\Delta x$. This method of procedure is illustrated in the following sections.

The integral sign originated from the old-fashioned long s, which is the initial letter of *summa*. Thus the integral was first conceived as a definite integral, the limit of a sum.

51. Plane Area.

In this section we shall illustrate the more general problems of finding plane area. In each case, the required area is divided into n vertical or horizontal *increments* ΔS. A rectangular *element of area* dS is found which

FIG. 37

arbitrarily closely approximates this increment by increasing the value of n. An approximation to the desired area is represented by the sum of all such elements of the area. The required area is the limit of this sum as the number of the elements is increased without limit. By means of the fundamental theorem, the limit of the sum is replaced by the definite integral and the solution is obtained by the evaluation of the definite integral.

First, we find the area bounded by a curve and a line;

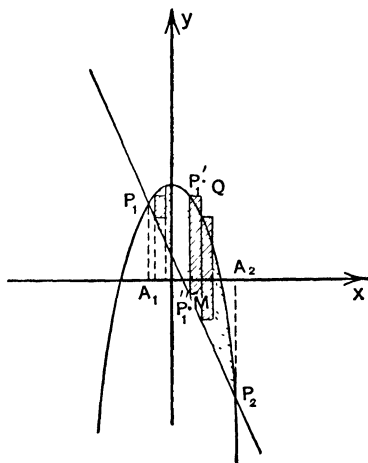$$y = 4 - x^2 \quad \text{and} \quad y = 1 - 2x.$$

The coordinates of the points of intersection of the parabola and the line are $P_1(-1,3)$ and $P_2(3,-5)$ as shown in Figure 37. The line segment A_1A_2 is divided into n equal parts

each of length Δx. The vertical lines from the line to the parabola divide the area S into n increments ΔS. If $P_i'(x_i, y')$ is a point on the parabola and $P_i''(x_i, y'')$ is a point on the line, then the length of the vertical line $P_i'' P_i'$ is

$$y_i' - y_i'',$$

which is positive for any value of x_i in the interval $-1 < x < 3$. An element of area, which approximates an increment of area ΔS, is

$$dS = (y_i' - y_i'') \Delta x.$$

In the figure one such element is the inscribed rectangle $P_i'' M Q P_i'$. From the given equations

$$dS = (3 + 2x_i - x_i^2) \Delta x.$$

If we sum the n elements corresponding to the n increments of the area, an approximation to the required area is found,

$$\sum_{i=1}^{i=n} (3 + 2x_i - x_i^2) \Delta x.$$

The area is the limit of this sum as n becomes infinite. Thus,

$$S = \lim_{\Delta x \to 0} \sum_{i=1}^{i=n} (3 + 2x_i - x_i^2) \Delta x.$$

By the fundamental theorem,

$$S = \int_{-1}^{3} (3 + 2x - x^2) \, dx = \frac{32}{3}.$$

Second, we find the area bounded by the curves

$$5y^2 = 4x \quad \text{and} \quad y^2 = 9 - x.$$

The coordinates of the points of intersection of the parabolas are $(5,2)$ and $(5,-2)$ as shown in Figure 38. Since both curves are symmetrical with respect to the x-axis, the required area is divided symmetrically by it and we need to find the area in the first quadrant only. In this problem the area is divided into n horizontal strips by dividing the line segment OB into parts each of length Δy. Since the length of the horizontal line segment $P'P''$ is $x_i'' - x_i'$ and is positive for any value $0 < y_i < 2$, the area of an element is

$$dS = (x_i'' - x_i') \Delta y = \frac{9}{4} (4 - y_i^2) \Delta y.$$

An approximation to the area is the sum of all such elements for any value of n. The area is the limit of this sum as n becomes infinite. Thus,

$$S = \lim_{\Delta y \to 0} \sum_{i=1}^{i=n} \frac{9}{4} (4 - y_i^2) \, \Delta y.$$

Hence, by the fundamental theorem,

$$S = \frac{9}{4} \int_0^2 (4 - y^2) \, dy = 12,$$
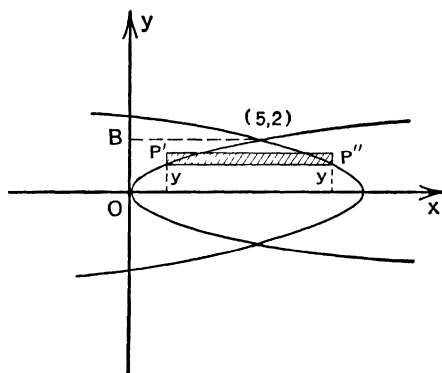
and the entire area enclosed is 24 square units.

Fig. 38

The solutions given for the two problems above illustrate the necessity of having precise geometric information concerning the curves under consideration. In the first, vertical elements were necessary to avoid the evaluation of two integrals. In the second, horizontal elements were similarly necessary. However, in some instances it is necessary to evaluate two integrals. An illustration of such a case is given below.
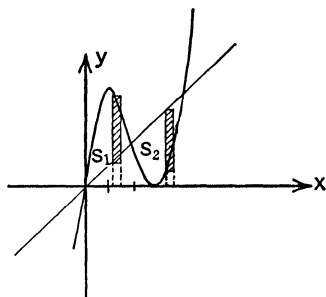

Fig. 39

We find the area which is enclosed by the curve and line:

$$y = x^3 - 6x^2 + 9x \quad \text{and} \quad y = x.$$

The coordinates of the points of intersection of the two loci are $(0,0)$, $(2,2)$ and $(4,4)$ as shown in Figure 39. The required area is divided into two parts and each must be computed sepa-

rately. The elements of area for the first and the second parts, by taking vertical strips, are

$$dS_1 = [(x_i{}^3 - 6x_i{}^2 + 9x_i) - x_i] \, \Delta x$$

and
$$dS_2 = [x_i - (x_i{}^3 - 6x_i{}^2 + 9x_i)] \, \Delta x.$$

Summing each of the two parts, taking the limit of each of the sums as Δx approaches zero and applying the fundamental theorem, we have

$$S = S_1 + S_2 = \int_0^2 (x^3 - 6x^2 + 8x) \, dx - \int_2^4 (x^3 - 6x^2 + 8x) \, dx$$

$$S = \left[\frac{x^4}{4} - 2x^3 + 4x^2\right]_0^2 - \left[\frac{x^4}{4} - 2x^3 + 4x^2\right]_2^4 = 4 - (-4) = 8.$$

Exercise 32

GROUP A

In solving each of the following problems, draw a careful figure, draw a representative element of the required area, express the approximate area, apply the fundamental theorem and find the area bounded as indicated.

1. The area enclosed by $y^2 = 16x$ and the line through the focus perpendicular to the axis.
2. The area enclosed by $x^2 = 2y$ and the line $y = 8$.
3. The area bounded by $y = 4x - x^2$ and $y = x$.
4. The area bounded by $y = x^2 - 4x$ and $y = x$.

Find the area enclosed by the following loci.

5. $y = x^3 - x^2 - 4x + 4$ and the x-axis.
6. $x^2 + 3y - 7 = 0$ and $2x + 3y - 4 = 0$.
7. $y^2 = x^3$ and $y^2 = 4x$.
8. $x^{1/2} + y^{1/2} = 2$ and $x + y = 4$.

GROUP B

9. Find the area between the curves $16x^2 = 9y$ and $9y^2 = 16x$ by taking the element in two different ways.
10. Find the area between $y = x^2(4 - x)$ and $x + y = 4$.
11. Find the area between $4y - x^2 - 4x - 4 = 0$ and $y + x^2 + 4x = 1$.
12. Find the area in the first and second quadrants which is bounded by $y^2 + 4x = 16$, $2y = x + 1$ and the x-axis.
13. Find the area bounded by $y^2 = 2x + 9$ and $x + y^2 = 0$.
14. Find the area between the curve $y = (x - 1)^2(x - 3)$ and the x-axis and find the ratio in which it is divided by the minimum ordinate.
15. Find the area enclosed by the curve $x = t^2 - 4$, $y = 2t$ and the y-axis.

16. Find the parametric equations of the path of a particle which moves so that the x- and the y-components of its acceleration are $2t$ and $3t^2$, respectively, if the initial position is $(2,1)$ and the initial velocity is such that its x- and y-components are 3 and 4, respectively.

52. Volume of a Solid of Revolution.

A plane curve which is rotated about a line lying in its plane, generates the surface of a *solid of revolution*. If a plane area is rotated about such a line, it generates the *volume* of a solid of revolution. The line is known as the *axis of revolution*.

In this section we shall consider two types of solids of revolution, those which are generated by the rotation of an area about a linear boundary as axis and those which are generated by the rotation of an area about an axis not bounding the area.
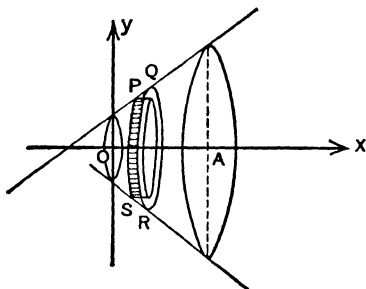

Fɪɢ. 40

As in the case of plane areas, the volume of a solid of revolution is found by forming increments of the volume, by approximating those increments by *elements of volume*, by summing the elements as the number of them becomes infinite, by applying the fundamental theorem and by evaluating the definite integral.

Let us find the volume of the solid generated by revolving the area between the line $y = 2x + 1$ and the x-axis from $x = 0$ to $x = 2$ about the x-axis.

The volume V generated is divided into elements by dividing the segment OA in Figure 40 into n parts each Δx in length and by passing planes perpendicular to the x-axis at each of the points of division. In the figure one of these elements is the truncated cone $PQRS$. Its volume is

$$\Delta V = \frac{\pi}{2}\left[y_i{}^2 + (y_i + \Delta y)^2 \right] \Delta x$$

or
$$\Delta V = \pi(y_i{}^2 \Delta x + 2y_i \overline{\Delta x}^2 + 2\overline{\Delta x}^3).$$

Taking the principal part of this infinitesimal only, we have the element of volume

$$dV = \pi y_i{}^2 \Delta x = \pi(4x_i{}^2 + 4x_i + 1) \Delta x.$$

This element of volume is the volume of an inscribed cylinder whose radius

is y_i and whose altitude is Δx.   Following the usual procedure,

$$V = \pi \int_0^2 (4x^2 + 4x + 1)\, dx = \frac{62}{3}\, \pi \text{ cu. units.}$$

We turn now to the problem of determining an element of volume which arbitrarily closely approximates the volume of an increment.   There are three types which we shall find useful, the *circular disc*, the *cylindrical shell* and the *circular ring*.   Each type is illustrated below.

Circular Disc Element.   If a narrow rectangle of width w and length h is rotated about one of its ends, it generates a thin circular cylinder whose volume is $\pi h^2 w$.   Such a volume is a *circular disc* element which may be used when the axis of revolution is one boundary of the area revolved.

Let us find the volume of the solid generated by the revolution of the area in the first quadrant bounded by the curve $y^2 = 2x$, the x-axis and the line $x - 8 = 0$ about the x-axis.   We shall be concerned with but one branch of the curve, namely $y = +\sqrt{2x}$.

The line segment OA in Figure 41 is divided into n equal segments of length Δx.   At each division, ordinates of the curve are drawn and either inscribed or circumscribed rectangles to the area are constructed.   Each rectangle, as $ABPC$,
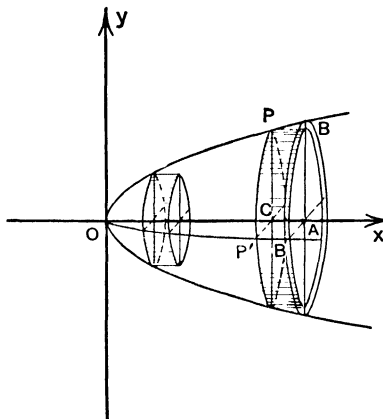

Fig. 41

will generate in its rotation about the x-axis, a circular disc whose volume is

$$dV = \pi y_i^2\, \Delta x.$$

An approximation to the volume is the sum of all such elements and the volume is

$$V = 2\pi \lim_{\Delta x \to 0} \sum_{i=1}^{i=n} x_i\, \Delta x.$$

By the fundamental theorem,

$$V = 2\pi \int_0^8 x\, dx = 64\pi.$$

If the same area as above, is rotated about the line $x - 8 = 0$ as an axis, a totally different solid of revolution is generated.   In this case a

point P on the curve describes a circle lying in a plane perpendicular to the y-axis, having its center on the line $x - 8 = 0$ and having a radius equal to $(8 - x_i)$. Hence the area of the circular cross section of the solid is $\pi(8 - x_i)^2$. The elements of volume are the circular discs of thickness Δy, one of which is drawn in Figure 42. Therefore, we write
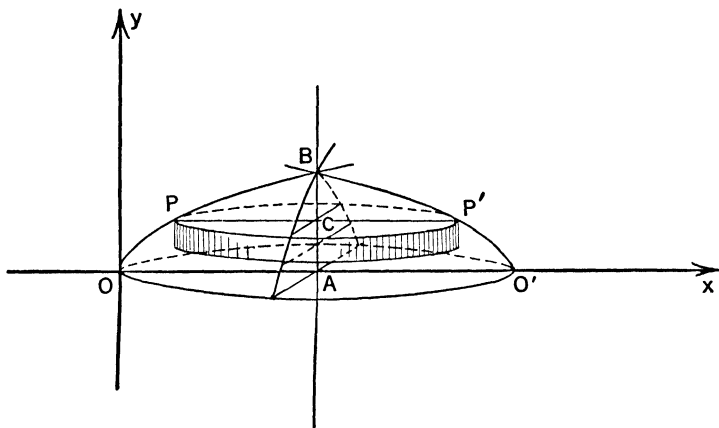
$$dV = \pi(8 - x_i)^2\,\Delta y.$$

Fig. 42

An approximation to the volume for any value of n is

$$\pi \sum_{i=1}^{i=n} \left(8 - \frac{y_i^2}{2}\right)^2 \Delta y.$$

Taking the limit and applying the fundamental theorem,

$$V = \frac{\pi}{4} \int_0^4 (16 - y^2)^2\,dy = \frac{2048}{15}\,\pi.$$

Cylindrical Shell Element. If a narrow rectangle of width w and length h is rotated about a line in its plane which is parallel to the longer sides and is at a distance of r from the nearer side, it generates a *cylindrical shell* whose inner radius is r, whose outer radius is $r + w$ and whose height is h. The volume of such a shell is

$$\pi\left[(r + w)^2 - r^2\right] h = \pi(2rw + w^2)h.$$

If w is an infinitesimal, the approximate volume of such a cylindrical shell for small values of w is the principal infinitesimal of the function, $2\pi rwh$.

The use of the cylindrical shell element of volume in finding volumes of revolution is frequently advantageous.

Let us find the volume generated by the rotation of the area in the first quadrant bounded by $y^2 = 4x$, the y-axis and the line $y - 8 = 0$ about the x-axis.
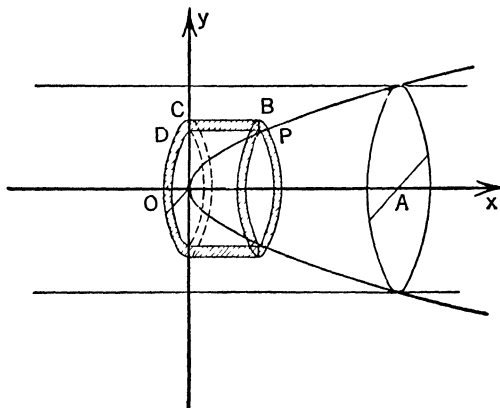

FIG. 43

The rectangle $PBCD$ in Figure 43 has the length x and the width Δy. When such a rectangle is rotated about the x-axis a cylindrical shell is formed whose inner radius is y, whose length is x and whose thickness is Δy. From above, the approximate volume of the element is

$$dV = 2\pi x_i y_i \, \Delta y.$$

The required volume is approximated by taking the sum of such elements,

$$\frac{\pi}{2} \sum_{i=1}^{i=n} y_i^3 \, \Delta y.$$

Taking the limit of this sum and applying the fundamental theorem,

$$V = \frac{\pi}{2} \int_0^8 y^3 \, dy = 512\pi.$$

Circular Ring Element. If a narrow rectangle of width w is rotated about a line in its plane which is parallel to the shorter sides and which is a distance of r_1 from the nearer side and r_2 from the farther side, it generates a thin *circular ring* whose volume is $\pi(r_2^2 - r_1^2)w$.

In application of the use of the circular ring element, let us find the volume generated by the area in the first quadrant between $y^2 = 2x$, the x-axis and $x - 8 = 0$ rotated about the y-axis.

In Figure 44 the rectangle $PQRS$ has the width Δy and $CP = x$ while $CQ = 8$. When this rectangle is rotated about the y-axis a circular ring is formed which is an element of the desired volume. We may think of
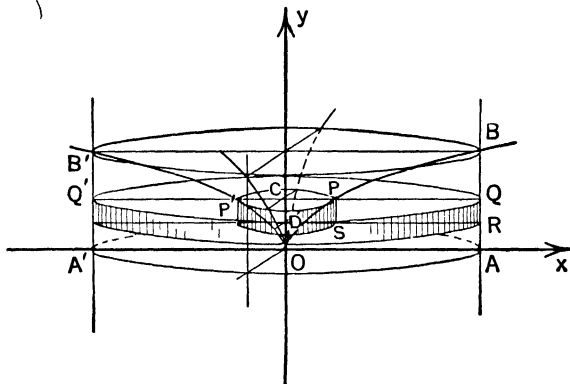

FIG. 44

the rectangle $CDRQ$ as rotated about the y-axis generating the circular disc and the rectangle $CDSP$ also rotated about the y-axis, the latter cutting the hole out of the circular disc thus forming the circular ring. The volume of the element is

$$dV = \pi(64 - x_i{}^2)\,\Delta y.$$

The approximate value of the volume is

$$\frac{\pi}{4} \sum_{i=1}^{i=n} (256 - y_i{}^4)\,\Delta y.$$

Taking the limit and applying the fundamental theorem,

$$V = \frac{\pi}{4} \int_0^4 (256 - y^4)\,dy = \frac{1024\pi}{5} \text{ cu. units.}$$

Exercise 33

GROUP A

In each of the following problems draw a good figure showing a representative element of the volume and find the volume which is generated under each of the following conditions.

1. The area in the first quadrant bounded by $x^2 = 4y$ and $y = 6$ is rotated about the y-axis.
2. The area bounded by $y = 2x - x^2$ and the x-axis is rotated about the x-axis.

3. The area in the first quadrant bounded by $x^2 = 8y$, the x-axis and $x = 2$ is rotated about the y-axis. Use cylindrical shell elements.

4. The area in the first quadrant bounded by $y = x^2$, $x + y = 2$ and the x-axis is rotated about the y-axis. Use circular ring elements.

Find the volumes which are generated by the rotation of the areas given.

5. Bounded by $y = 2x^2 + x^3$ and the x-axis rotated about the x-axis.

6. Bounded by $y^2 = 8x$ and $x = 2$ rotated about $x = 2$.

7. Bounded by $y^2 = x$ and $x = 4$ rotated about $x = 6$.

8. In the first quadrant bounded by $x^2 = 2y$, $x + y = 4$ and the x-axis rotated about the x-axis. Use cylindrical shell elements.

GROUP B

Find the volumes which are generated by the rotation of the areas given.

9. Bounded by $y = 2x^2$, the y-axis, $x = 3$ and $y + 2 = 0$ rotated about $y + 2 = 0$.

10. Bounded by $y^2 + 4x = 0$, the x-axis, $x = 2$ and $y = 3$ rotated about $x = 2$.

11. Bounded by $y = 4x - x^2$, and $y = x^2 - 4x + 6$ rotated about the x-axis.

12. Bounded by $y^2 = 4x$, $x + y - 8 = 0$ and the y-axis rotated about the y-axis.

13. Bounded by $2y^2 = x^3$ and $x = 2$ rotated about $x = 2$.

14. Find the area between $x^2 + 2x + y = 2$ and $3x^2 + 6x - 4y = 48$.

15. A particle moves on a line so that its velocity at any time is given by $t^2 - 3t + 2$. Show that its direction of motion changes at the end of the second unit interval of time. Find how far it moves during the second two unit interval of time. Draw a figure for the given function and show the geometric illustration of the latter solution.

16. A wire $2a$ ft. long is to be bent into the form of an isosceles triangle and one-half of it is to be revolved about the altitude to form a cone of revolution. Find the base and altitude of the triangle which will give a cone of maximum volume.

GROUP C

Derive formulas for the volume of each of the following solids by integration.

17. A right circular cone of altitude h and radius of its base r.

18. A frustum of a right circular cone of altitude h and radii of its bases r_1 and r_2.

19. A sphere of radius r.

20. A cylinder of altitude h and radius of its base r.

21. A segment of a sphere of altitude h and radii of its bases r_1 and r_2.

22. Find the volume generated by the rotation of the area between $x^{1/2} + y^{1/2} = 2$, the x-axis and the y-axis about the y-axis.

23. Find the volume generated by the rotation of the area in first and second quadrants between $x^{2/3} + y^{2/3} = 4$ and the x-axis about the x-axis.

24. Show that the volume of a thin spherical shell of radius r and thickness Δr is approximately $4\pi r^2 \, \Delta r$. Using such spherical shell elements, find the volume of a sphere of radius a.

53. Volumes of Miscellaneous Solids.

In finding the volumes of miscellaneous solids it shall be assumed that the area of a cross section is known. A *cross section* of a solid is a closed area bounded by the intersection of the enveloping surface and a cutting plane. The data of the problem enable one to compute the area of a cross section which lies on any one of a series of parallel planes. In addition, it must be possible to express the perpendicular distance of any, such cross section from some fixed point. If the variable x represents that distance, then the area of the cross section multiplied by its thickness Δx is an element of the volume. It is seen that the various elements which were used in finding the volumes of solids of revolutions are special cases of the present more general problem.
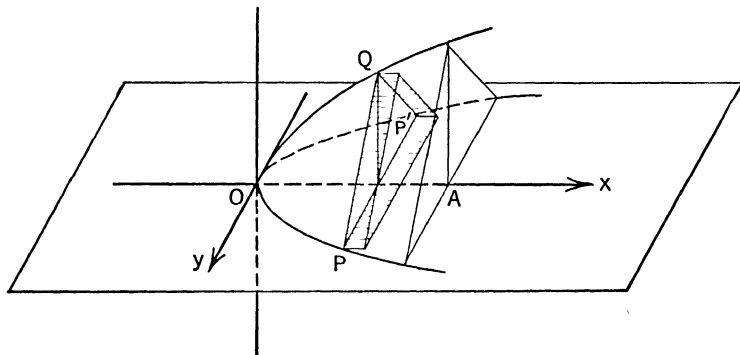

FIG. 45

Let us find the volume of the solid of which any cross section made by a plane perpendicular to the x-axis is an equilateral triangle having two of its vertices on the curve $y^2 = 4x$, if the altitude OA of the solid is 8.

Since one side of the triangle forming a cross section is $2y_i$, the area of the triangle $PP'Q$ in Figure 45 is $\sqrt{3}\ y_i^2$. Accordingly, the element of volume is

$$dV = \sqrt{3}\ y_i^2\ \Delta x.$$

Proceeding as in the previous illustrations,

$$\text{Approximate } V = \sum_{i=1}^{i=n} 4\sqrt{3}\ x_i\, dx$$

$$V = 4\sqrt{3} \int_0^8 x\, dx = 128\ \sqrt{3} \text{ cu. units.}$$

As a second illustration, we find the volume common to two right circular cylinders each of radius a whose axes intersect at right angles.

In Figure 46 one eighth of the common volume of the two cylinders is represented. The lines OA and OC represent the perpendicular axes intersecting at O. The arcs BD and BE are the arcs of circles on the surfaces of the cylinders which intersect in the curve BF. Let OA and OB be taken as the x- and y-axes and take any section perpendicular to the y-axis which is the square $PQRS$ whose side is x_1. Since

Fig. 46

its distance above the plane AOC is y_1, the element of volume is
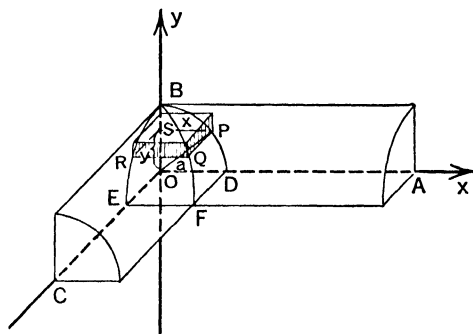
$$dV = x_1{}^2 \, \Delta y.$$

But since the radii of the cylinders are a,

$$x_1{}^2 + y_1{}^2 = a^2.$$

Hence,
$$V = 8 \int_0^a (a^2 - y^2)\, dy = \frac{16}{3} a^3.$$

Exercise 34

GROUP A

A solid has a circular base of radius 6 ins. and the diameter of this base is the line OA. Find the volume of each of the following solids where every plane section of the solid perpendicular to OA is given as follows.

1. An equilateral triangle.
2. An isosceles right triangle with its hypotenuse in the plane of the base of the solid.
3. An isosceles right triangle with one of its equal sides in the plane of the base of the solid.
4. An isosceles triangle with its altitude equal to its base, the latter being in the plane of the base of the solid.

A solid has an elliptical base whose major axis is 12 ins. long and whose minor axis is 6 ins. long. Find the volume of each of the following solids where every plane section of the solid perpendicular to the major axis of the base is given as follows.

5. A square.
6. An equilateral triangle.
7. An isosceles triangle with its altitude equal to its base, the latter being in the plane of the base of the solid.

A solid has as base a segment of a parabola which is cut off by a chord 24 ins. long, 9 ins. from the vertex of the parabola and perpendicular to the axis.   Find the volume of each of the following solids where every plane section of the solid perpendicular to the axis of the parabolic base is given as follows.

8. A square

9. An equilateral triangle.

10. An isosceles triangle with altitude 4 ins. whose base is in the base of the solid.

GROUP B

11. Any section of a solid made by a plane perpendicular to a line segment OA is a circle, tangent to OA, whose center is on a line segment OB making an angle whose tangent is 2 with OA If the altitude of the solid along OA is 9 ins, find the volume.

12. On a spherical ball of wood two great circles are drawn intersecting at right angles at the points P and Q.   The radius of the ball is a ins.   If the wood is cut away so that any cross section perpendicular to PQ is a square with its vertices on the two great circles, find the volume remaining.

13. Any plane section of a solid made by a plane perpendicular to the r-axis is a square of which the center lies on the x-axis and two opposite vertices lie on the curve $y^2 = 8x$.   Find the volume of the solid if the altitude along the x-axis is 10 ins.

14. Any plane section of a solid made by a plane perpendicular to the y-axis is a circle with the ends of one of its diameters on the curves $y^2 = x$ and $y^2 = 2x - 4$. Find the volume of the solid between the points of intersection of the curves.

15. The center of a square moves along the diameter of a circle of radius a with the plane of the square perpendicular to the diameter of the circle and two opposite vertices of the square move on the circumference of the circle.   Find the volume of the solid generated.

16. A variable equilateral triangle moves with its plane perpendicular to the y-axis and the ends of its base on the curves $x^2 = 16ay$ and $x^2 = 4ay$, and to the right of the y-axis.   Find the volume generated by the triangle as it moves from the origin to $y = a$.

17. A variable rectangle moves from a point O so that one side is equal to its distance from O and the other is equal to the square root of this distance   Find the volume of the solid generated if the rectangle moves a distance from O of 4 ft.

18. Find the volume of a wedge cut from a right circular cylinder of radius a by a plane intersecting the base in a diameter and inclined to it at an angle of 60°.

19. Derive the formula for the volume of a right pyramid of altitude h and having a square base of side a, using integration.

20. Derive the formula for the volume of the frustum of a right pyramid of altitude h and having square bases of sides a_1 and a_2, using integration.

21. The carrying capacity of a rectangular beam varies as the product of its width and the cube of its depth.   Find the dimensions of the beam having maximum carrying capacity which can be cut from a circular cylindrical log of radius a.

22. If water is flowing from a right circular conical tank, 6 ft. across the top and 4 ft. deep, at the rate of 2 cu. ft. per min., find the rate at which the surface of water is falling and find the rate at which the circumference of the surface of the water in moving down the side of the cone when the depth is 2 ft.

54. Fluid Pressure.

A liquid which is at rest exerts a force on a surface in contact with it. In this section we shall consider the problem of finding this force in special cases where the submerged surface is taken as a vertical bounded area lying in a plane.

Assume an increment of area ΔS containing a fixed point A so that as ΔS approaches zero the point A is in its interior. Let ΔF represent the force exerted by the liquid on one side of the area ΔS. Then the *average pressure* is $\Delta F/\Delta S$, which is the average force per unit of area ΔS. The *pressure* at the fixed point A is defined as

$$P = \lim_{\Delta S \to 0} \frac{\Delta F}{\Delta S}.$$

In the study of hydrostatics it is shown that the pressure exerted at a point in a liquid is proportional to the depth of the point below the surface. Thus,

$$P = hw,$$

where h is the depth of the point A and w is weight of the liquid per unit volume. But since the pressure is the same at all points having the same depth h, the pressure is taken to be the force exerted by the liquid on each unit of this area.

Suppose that the area $CDEF$ in Figure 47 is vertically submerged in a liquid with horizontal top CD at a depth of $AC = a$ below the surface AB of the liquid. Let the area be divided into n *horizontal* strips equally spaced along the vertical line CF. If the upper edge of any strip has the distance y_i below AB, if the width of the strip is Δy and if its length is x, then
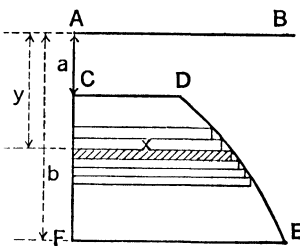
$$dS = x_i \, \Delta y,$$

Fig. 47

which approximates the element of area ΔS_i. The pressure at any point on the upper edge of ΔS_i is wy_i and the pressure at any point on the lower edge of ΔS_i is $w(y_i + \Delta y)$. Then the pressures at points between the two, range between these, or in terms of fluid force,

$$wy_i \, \Delta S_i < \Delta F_i < w(y_i + \Delta y) \, \Delta S_i$$

and

$$wy_i dS_i < \Delta F_i < w(y_i + \Delta y) \, dS_i.$$

Hence, ΔF_i differs from $wy_i dS_i$ by an infinitesimal of higher order than it.

Using the principal part of ΔF,

$$dF = wy_i dS_i = wx_iy_i\, \Delta y.$$

Therefore,

$$F = \lim_{\Delta y \to 0} \sum_{i=1}^{i=n} wx_iy_i\, \Delta y$$

and

$$F = w \int_a^b xy\, dy,$$

where the limits a and b are taken so that $b - a$ is the width CF of the area.

Let us find the horizontal force against a vertical triangular water gate whose horizontal base is 12 feet long and 3 feet below the surface of the water and whose altitude is 10 feet, where the vertex is below the base.
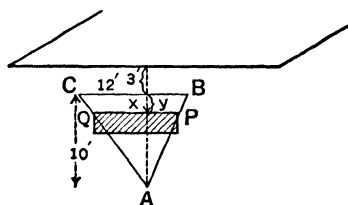

Fig. 48

In Figure 48 the data of the problem are represented and a horizontal rectangle is circumscribed to a strip of the area. Let the length of such a rectangle be x and let the upper base be a distance of y below the base of the given triangle, where y is directed downward as positive. Then the element of area is

$$dS = x_i\, \Delta y.$$

The distance below the surface of the water of any element of area is $y_i + 3$. If we denote the approximate force on the rectangular element of area by dF, we have what may be called an *element of force*,

$$dF = wx_i(y_i + 3)\, \Delta y.$$

Summing all such elements, an approximation to the force is

$$w \sum_{i=1}^{i=n} x_i(y_i + 3)\, \Delta y.$$

The force is the limit of this sum, from which

$$F = w \int_0^{10} x(y + 3)\, dy.$$

From the similar triangles ABC and APQ,

$$5x + 6y = 60.$$

Solving for x and substituting in the integral,

$$F = \frac{6}{5} w \int_0^{10} (30 + 7y - y^2)\, dy = 380w.$$

If w represents the weight of a cubic foot of water and $w = 62\frac{1}{4}$ lbs.,

$$F = 23{,}655 \text{ lbs.} = 11.83 \text{ tons.}$$

Similarly, let us find the force exerted by a liquid on a vertical parabolic segment whose altitude is a and whose horizontal base is $2b$, where the vertex is in the surface of the liquid.
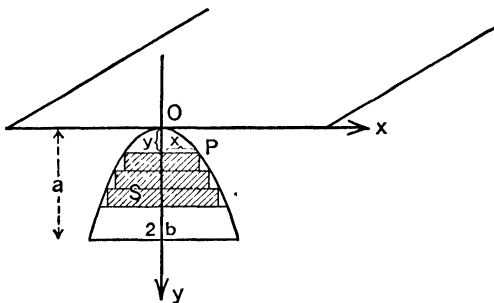

FIG. 49

Taking the x-axis in the surface of the liquid and the y-axis through the vertex, downward as positive, the equation of the parabola is

$$ax^2 = b^2y.$$

The element of area for an inscribed rectangle as shown in Figure 49 is

$$dS = 2x_1\, \Delta y$$

and the element of force is

$$dF = 2wxy_1\, \Delta y.$$

Hence, $$F = 2w \int_0^a xy\, dy$$

$$F = \frac{2b}{\sqrt{a}} w \int_0^a y^{3/2}\, dy = \frac{4}{5} a^2 bw.$$

Exercise 35

GROUP A

A triangular water dam has a base 10 ft. and altitude 8 ft. and is submerged in water with its altitude vertical. Find the force exerted on the dam under each of the following conditions.

1. The vertex is in the surface of the water with base below it.
2. The vertex is 3 ft below the surface and the base is below the vertex.
3. The base is in the surface of the water with the vertex below it.
4. The base is 4 ft below the surface and the vertex is below the base.
5. In Problem 3 find the position of a horizontal line so that the force above it is equal to that below it.

A trapezoidal water dam has its horizontal bases 50 ft. and 60 ft. long and has an altitude of 20 ft. and is submerged in water with its altitude vertical. Find the force exerted on the dam under each of the following conditions.

6. The longer base is in the surface of the water with the shorter base below it.
7. The longer base is 2 ft. above the surface and the shorter base is below it.
8. The shorter base is in the surface of the water with the longer base below it.
9. The shorter base is 5 ft. below the surface and the longer base is below the shorter one.
10. In Problem 6 find the position of a horizontal line so that the force above it is equal to that below it.

GROUP B

A parabolic water gate is formed from a parabola by a chord 32 ft long at a distance of 16 ft. from the vertex and perpendicular to the axis. If the water gate is submerged in water with its axis vertical, find the force exerted against it under each of the following conditions.

11. The vertex is in the surface of the water with the base below it.
12. The vertex is 4 ft. above the surface with the base below it.
13. The vertex is 3 ft below the surface with the base below it.
14. The base is in the surface of the water with the vertex below it.
15. The base is 4 ft. below the surface with the vertex below the base.
16. The base is 5 ft above the surface of the water with the vertex below it
17. In Problem 11 find the position of a horizontal line so that the force above it is equal to that below it.

18. A board 5 ft. square is submerged in water with one vertex in the surface and with the diagonal through that vertex vertical Find the force exerted on the board both above and below the second diagonal

19. In accordance with Boyle's Law $PV = k$, a volume of air varies inversely with the pressure, where k is a constant. If the pressure is decreasing at the rate of 4 lbs. per square inch per minute, find the corresponding rate of change of the volume at the instant the pressure is 100 lb. per in. and the volume is 20 cu ins

20. Find and classify the critical points of the curve $y = 3x^5 - 15x^4 + 10x^3 + 30x^2 - 45x - 4$. Find the abscissas of the inflection points and give the intervals over which the curve is concave upward and downward.

55. Work.

If a force of constant magnitude is applied to a particle in a fixed direction and if the particle is moved along a line in the same direction, the force is said to do *work*. From physics

$$W = F \cdot x,$$

where W is the work done by the force F in moving a particle the distance x. If the force is expressed in pounds and the distance in feet, the work done is in *foot-pounds*.

We shall be concerned with problems in which the forces acting on a body are variable and are functions of the distance over which the body is moved. Hence,

$$F = f(x),$$

where $f(x)$ is a positive continuous function of the variable distance x of the moving body from a fixed point in the line of motion.

Let us consider a particle which is moved by a force $f(x)$ from A to B along the line OP as shown in Figure 50, where

$$OA = a, \quad OB = b \quad \text{and} \quad OP = x.$$

The line segment AB is divided into n equal segments Δx. At any point P_1, where $OP_1 = x_1$, $f(x_1)$ represents the force exerted on the particle. Then by the definition of work, the work done by the force over the interval Δx is approximately,
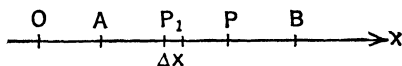
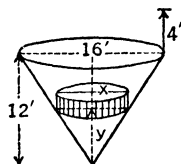$$dW = f(x_i)\, \Delta x.$$

FIG 50

FIG. 51

For any interval, we shall call this an *element of work*. The sum of all such elements forms an approximation to the total work in moving the particle over the interval from $x = a$ to $x = b$. Hence,

$$W = \int_a^b f(x)\, dx.$$

When work is done against the force of gravity, the force exerted is the *weight* of the body and the distance is the *vertical* distance through which the body is lifted.

A conical tank 16 ft. across the top and 12 ft. deep is represented in Figure 51. If the tank is full of water, we wish to find the work necessary to pump the water to a height of 4 ft. above the top of the tank.

The volume of any inscribed circular disc element of the given volume is

$$dV = \pi x_i^2\, \Delta y$$

and the weight of the element of volume filled with water is

$$\pi w x_i^2 \, \Delta y,$$

where w represents the number of pounds in a cubic foot of water. This function is the force against which the work is exerted. The distance to which any element of volume filled with water must be lifted is $16 - y_i$. Hence, the element of work is

$$dW = \pi w x_i^2 \, (16 - y_i) \, \Delta y.$$

From similar triangles, we find that $2y = 3x$. Substituting the value of x and taking the sum of all the elements of work,

$$W = \frac{4}{9} \, \pi w \lim_{\Delta y \to 0} \sum_{i=1}^{i=n} (16y_i^2 - y_i^3) \, \Delta y$$

$$W = \frac{4}{9} \, \pi w \int_0^{12} (16y^2 - y^3) \, dy = 1792 \, \pi w \text{ ft. lbs.}$$

Exercise 36

GROUP A

1. The force in pounds acting on a body is $F = 2x - 5$, where x is the distance of the body from the source of the force. Find the work done in moving the body from $x = 3$ to $x = 5$.
2. The force in pounds acting on a body is $F = 3x^2 + 1$, where x is the distance of the body from the source of the force. Find the work done in moving the body from a point where $F = 13$ to $F = 28$ lbs.
3. If the weight of a body varies inversely as the square of its distance from the center of the earth, find the work done in lifting B lbs from the surface of the earth to a height of a miles above the surface. Use 4000 miles as the radius of the earth.
4. A positive charge k of electricity at O repels a unit positive charge at a distance x from O with the force k/x^2. Find the work done in carrying the unit charge from $x = 2a$ to $x = a$.
5. The force necessary to stretch a spring is proportional to the amount the spring is stretched. If a force of 3 lbs. will stretch a spring 6 ins., find the work done in stretching it 3 ins. and in stretching it 12 ins.
6. A vertical cylindrical water tank has a radius of 3 ft. and a height of 10 ft. If the tank is full of water, find the work done in pumping the water to a height of 6 ft. above the top.
7. If the water tank in Problem 6 stands on a horizontal roof 50 ft. above the ground, find the work done in pumping the tank full of water from the ground through a pipe which delivers at the top of the tank.
8. A particle moves on a straight line so that its distance from O at any time is $x = bt^2$. If the resistance of the air is proportional to the velocity, find the work done against that resistance as the body moves from $x = 0$ to $x = 9$.

GROUP B

9. A conical tank of height 12 ft. has a circular base of radius 6 ft. If it is full of water, find the work done in pumping the water to a height of 3 ft. above the vertex until the surface is lowered 6 ft.

10. A segment of a parabola with a vertical axis is revolved about its axis to form a tank 6 ft. deep and 4 ft. across the top. If the water in the tank has a depth of 4 ft., find the work done in pumping the water to the top of the tank.

11. A segment of a parabola with a vertical axis is revolved about its axis to form a tank 6 ft. deep and 4 ft. across the base. If the base of the tank is 10 ft. from the ground, find the work done in pumping the tank full of water through a pipe from the ground to the bottom of the tank.

12. Find the work done in pumping the water from a hemispherical tank of radius 6 ft. to a height of 5 ft. above the top of the tank.

13. A water tank has the form of the frustum of a right circular cone. If the tank is 6 ft. deep, 8 ft. across the top and 10 ft. across the bottom, find the work done in emptying the tank over the top of the tank.

14. A rectangular tank 6 ft. deep, 5 ft. across and 10 ft. long stands with its base 12 ft. from the ground. Find the depth of the water in the tank when one-half of the necessary work has been done to fill the tank from the ground through a pipe in the bottom.

15. Find approximately, the value of $\dfrac{1}{\sqrt{10 - x}}$ for $x = 1.001$. Find the percentage error of the approximation.

16. The motions of two particles moving on a line are given by the equations $s_1 = 2t^3 - 5t^2 + 2t - 20$ and $s_2 = t^4 + 4t^2 - 10t - 2$. Find the velocities and the positions of the two particles at the instant they have equal accelerations.

17. Given the equation $y = \dfrac{1 - x}{1 + x}$. Are the first three derivatives continuous for all values of x? Does the curve possess an inflection point or a critical point? Find the interval of x in which the curve is concave upward.

18. Given $x = \dfrac{1 + t}{1 - t}, y = \dfrac{1 - t}{1 + t}$. Find $\dfrac{d^2y}{dx^2}$ and $\dfrac{d^3y}{dx^3}$.

19. A particle moves on the curve $y = x^2 + 2x + 3$ with the x-component of its velocity equal to 3. Find the y-component of the velocity and the x- and the y-components of the acceleration at any time.

20. If a stone is thrown from the top of a cliff 200 ft. high directly toward an object 100 ft. from the foot of the cliff with an initial velocity of 50 ft. per second, find the distance by which the stone misses the mark.

GROUP C

21. A ball rolls down an incline whose equation is $16y = 16 - x^2$ to the x-axis. If the horizontal component of the velocity is 6, find the speed of the ball when it reaches the x-axis.

22. Find the volume generated by rotating the curve $4x^2 + 9y^2 = 36$ about the x-axis. Find the portion of this volume which lies between the two planes perpendicular to the major axis of the curve at the two points of trisection.

23. A vertical rectangular water dam is 10 ft. wide and 6 ft. deep. Find the total force on the dam when the water level is 8 ft. above the top of the dam. Find how much higher the water must rise to double the force on the dam.

24. A hemispherical tank of diameter 20 ft. is full of oil weighing 60 lbs. per cu. ft. The oil is pumped to a height of 10 ft. above the top of the tank by an engine which can do work at the rate of 16,500 ft lbs per min. How long will it take to empty the tank?

25. A bucket of weight 50 lbs. is to be lifted from the bottom of a shaft 100 ft. deep. The weight of the rope used to hoist it is 0 5 lb. per ft. Find the work done.

26. A square hole with sides slanting at 45° is to be dug in the ground. The soil to be removed weighs 200 lbs. per cu. ft. If the top of the hole is to be 10 ft. on a side and if the depth is to be 3 ft., find the work done in excavating the soil and in lifting it to the level of a truck 3 ft. above the ground.

CHAPTER VII

TRIGONOMETRIC FUNCTIONS

56. Circular Measure of Angles.

In practical problems which arise in surveying and in related fields of study, where the solution of triangles is required, angles are more conveniently measured in degrees. However, in theoretical problems and those which arise in the study of the calculus and its applications, the radian measure of angles is more convenient.

Radian. A *radian* is the angle at the center of a circle which subtends a circular arc equal in length to the radius of that circle. As an immediate consequence of that definition

$$s = r \cdot \theta,$$

where s is the linear measure of the arc and r is the measure of the radius of the circle in the same units. Then θ is measured in radians, that is, θ represents the number of radians in the subtended angle.

From the elementary relation $c = 2\pi r$, where c is the linear measure of the circumference of a circle and r is that of the radius in the same units,

$$\frac{c}{r} = 2\pi \text{ radians.}$$

Hence, the entire angular magnitude about a point is 2π radians which is equivalent to 360°. Then

$$\pi \text{ radians} = 180°.$$

57. Graphs of Trigonometric Functions.

The six fundamental trigonometric functions are

$$\sin x, \quad \cos x, \quad \tan x, \quad \cot x, \quad \sec x, \quad \csc x.$$

If each of these functions is represented by y, there are six equations expressed in which x is the independent variable. The range of values of x is unlimited, both positively and negatively. While each function is a single-valued function, they are not all continuous for all values of x. The

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variations of the different functions differ very markedly and the range of values for some of the functions is limited.

Graph of $y = a \sin bx$. In drawing the graph of the function

$$y = 2 \sin 3x,$$

it is convenient to find first, values of x for which $y = 0$, those for which the sine has the maximum value $+1$, by letting $y = 2$, and those for which the sine has the minimum value -1, by letting $y = -2$. Thus:

If $y = 0,$ $x = \cdots,$ $-\frac{2}{3}\pi,$ $-\frac{1}{3}\pi,$ $0,$ $\frac{1}{3}\pi,$ $\frac{2}{3}\pi \cdots.$

If $y = 2,$ $x = \cdots,$ $-\frac{1}{2}\pi,$ $\frac{1}{6}\pi,$ $\frac{5}{6}\pi, \cdots.$

If $y = -2,$ $x = \cdots,$ $-\frac{5}{6}\pi,$ $-\frac{1}{6}\pi,$ $\frac{1}{2}\pi, \cdots.$
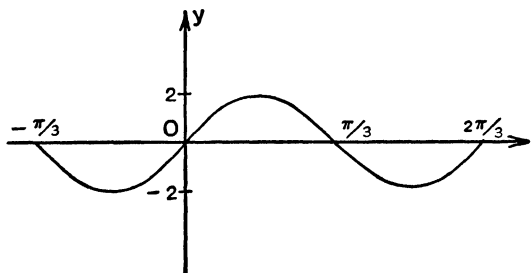

Fig. 52

The graph of the function is drawn in Figure 52 from $x = -\pi/3$ to $x = 2\pi/3$.

The *period* of a trigonometric function is the least angular magnitude after which the values of the function repeat themselves in the same order. The period of the function $a \sin bx$ is $2\pi/b$. The constant a is called the *amplitude* of the function.

Graph of $y = a \cos(bx + c)$. In drawing the graph of the function

$$y = 2 \cos (3x + 1),$$

we may proceed as above, solving for x if $y = 0$, $y = 2$ and $y = -2$. However, a more convenient method of attack is to use a transformation of axes. If we let

$$x' = x + \tfrac{1}{3}, \quad \text{then} \quad y = 2 \cos 3x'.$$

When this curve has been drawn with reference to a pair of x- and y'-axis as in Figure 53, the original

Fig. 53

y-axis is drawn parallel to the y'-axis and $\frac{1}{3}$ unit to the right of it.
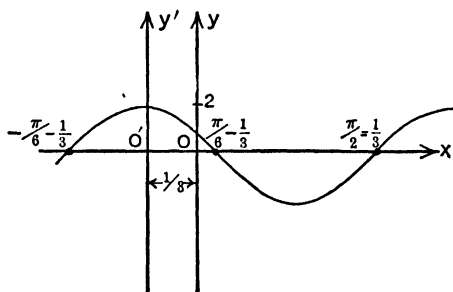
Graph of $y = a \tan bx$. The graph of the function

$$y = a \tan bx$$

is discontinuous for $x = n\pi/2b$, where n is a positive or a negative odd integer.   As x approaches any one of these values, y increases or decreases
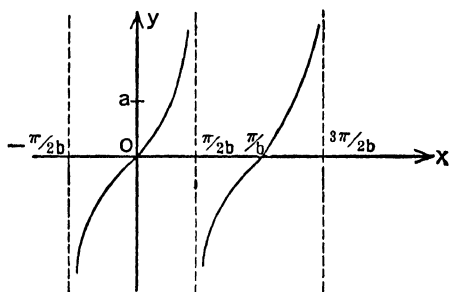

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without limit, that is, becomes infinite.   Hence, all lines having the equation $2bx - n\pi = 0$, for these values of n, are vertical asymptotes.   Two branches of the curve are drawn in Figure 54 in which it may be seen that the period of a tangent curve is π/b.

Exercise 37

GROUP A

Trace the curve for each of the following functions, indicating the vertical and horizontal scales used and giving the period of each function.

1. $y = \sin 3x$.

2. $y = \cos 4x$.

3. $y = 2 \tan 2x$.

4. $y = \sec x$.

5. $y = \csc x$.

6. $y = 2 \cot x$.

7. $y = 4 \sin \dfrac{x}{2}$.

8. $y = \cos \tfrac{2}{3}x$.

9. $y = 4 \tan \dfrac{x}{2}$.

10. $y = 2 \sec 3x$.

11. A flywheel of radius 20 ins. makes 10 revolutions per sec.   Find the linear speed of a point on the rim of the wheel and of a point on a spoke 5 ins. from the center.

12. Find the angle which a chord 6 ins. long subtends at the center of a circle of radius 20 ins.   Find the length of the smaller arc subtended.

GROUP B

Trace the graph of each of the following functions.

13. $y = 2 + \sin 2x$.

14. $y = 4 - \cos 3x$.

15. $y = 2 \sin \left(x + \dfrac{\pi}{2} \right)$.

16. $y = 3 \cos (x - \pi)$.

17. $y = \tan\left(x + \dfrac{\pi}{4}\right)$.   **20.** $y = \frac{1}{2}\tan(2x + 1)$.

18. $y = 2\sin(2x - \pi)$.   **21.** $y = \cos x + \sin x$.

19. $y = 3\cos(x - 1)$.   **22.** $y = x + \sin x$.

23. The angle between two radii of a circle of radius 6 ins. is 20°. Find the area of the sector between them and the area of the segment formed by the chord joining their extremities and the minor subtended arc

24. A horizontal cylindrical tank is 12 ft long and the circular ends have a radius of 2 ft. Find the volume of water in the tank if the water is 1 ft. deep, and if the water is 3 ft deep.

58. Evaluation of Two Limits.

If θ is an infinitesimal measured in radians, the variable $\sin\theta$ is of the same order. Thus

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = k.$$

We wish to prove that $k = 1$ for reasons which appear in the next section. In Figure 55 an angle 2θ is constructed, where
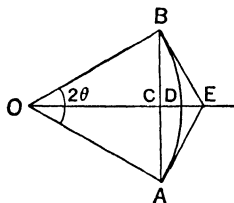
$$2\theta = \angle AOB.$$

With the vertex O as a center and with any length r as a radius, the circular arc ADB is drawn. At the points A and B, tangents to the circle are constructed which intersect at the point E lying on the bisector OD of the angle 2θ. The chord ACB is drawn. From the figure it is obvious that

Fig. 55

$$\text{Chord } ACB < \text{Arc } ADB < AE + EB.$$

Again from the figure,

$$ACB = 2AC = 2r\sin\theta,\ AE = EB = r\tan\theta.$$

$$ADB = 2r\theta, \text{ where } \theta \textit{ is measured in radians.}$$

Whence, $2r\sin\theta < 2r\theta < 2r\tan\theta,$

$$\sin\theta < \theta < \tan\theta.$$

Dividing the inequality by $\sin\theta$,

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}.$$

and inverting,

$$1 > \frac{\sin \theta}{\theta} > \cos \theta.$$

As θ approaches zero, $\cos \theta$ approaches 1. And since the ratio of the angle and its sine lies between 1 and $\cos \theta$, it also must approach 1. Thus it has been proved that: The limit of the ratio of an angle and its sine is unity as the angle approaches zero, *provided that the angle is measured in radians.*

If θ is an infinitesimal, the variable $1 - \cos \theta$ is an infinitesimal of higher order. Thus

$$\lim_{\theta \to 0} \frac{1 - \cos \theta}{\theta} = 0.$$

This statement is proved as follows:
From trigonometry

$$1 - \cos \theta = 2 \sin^2 \frac{\theta}{2}.$$

Dividing both sides of the equation by θ,

$$\frac{1 - \cos \theta}{\theta} = \frac{\sin^2 \dfrac{\theta}{2}}{\dfrac{\theta}{2}} = \sin \frac{\theta}{2} \cdot \frac{\sin \dfrac{\theta}{2}}{\dfrac{\theta}{2}}.$$

Taking the limit of both sides of the equation as θ approaches zero,

$$\lim_{\theta \to 0} \frac{1 - \cos \theta}{\theta} = \lim_{\theta \to 0} \sin \frac{\theta}{2} \cdot \lim_{\theta \to 0} \frac{\sin \dfrac{\theta}{2}}{\dfrac{\theta}{2}} = 0 \cdot 1 = 0.$$

59. Formulas for Differentiation of Trigonometric Functions.

The formulas for the differentiation of the six fundamental trigonometric functions are as follows, where u denotes any function of x which can be differentiated:

(8)
$$\frac{d}{dx} \sin u = \cos u \frac{du}{dx}.$$

(9)
$$\frac{d}{dx} \cos u = - \sin u \frac{du}{dx}.$$

(10)
$$\frac{d}{dx} \tan u = \sec^2 u \frac{du}{dx}.$$

(11)
$$\frac{d}{dx} \cot u = -\csc^2 u \frac{du}{dx}.$$

(12)
$$\frac{d}{dx} \sec u = \sec u \tan u \frac{du}{dx}.$$

(13)
$$\frac{d}{dx} \csc u = -\csc u \cot u \frac{du}{dx}.$$

60. Derivation of Trigonometric Differentiation Formulas.

The derivation of the derivative of the sine of a function requires the use of the delta process which was used in earlier chapters. However, once this derivative is found, the result can be used in the derivation of the derivatives of other trigonometric functions. Consequently, the first derivation is a fundamental one.

Derivative of sin u. Let

$$y = \sin u,$$

where u is any function of x which can be differentiated. If x is given the increment Δx, u and y will take on the corresponding increments Δu and Δy, respectively. Then

$$y + \Delta y = \sin(u + \Delta u),$$

$$\Delta y = \sin u \cos \Delta u + \sin \Delta u \cos u - \sin u$$

and
$$\Delta y = \cos u \sin \Delta u - (1 - \cos \Delta u) \sin u.$$

Dividing both sides of the equation by Δx,

$$\frac{\Delta y}{\Delta x} = \cos u \sin \Delta u \frac{1}{\Delta x} - (1 - \cos \Delta u) \sin u \frac{1}{\Delta x}.$$

The limits of the terms of this equation can be evaluated if Δu is supplied in both numerator and denominator of each term in the right hand member. Thus

$$\frac{\Delta y}{\Delta x} = \cos u \frac{\sin \Delta u}{\Delta u} \frac{\Delta u}{\Delta x} - \frac{1 - \cos \Delta u}{\Delta u} \sin u \frac{\Delta u}{\Delta x}.$$

Taking the limit of each term as Δx approaches zero by making use of the evaluations of the limits in Section 58, and by making use of the fact that Δu also approaches zero, we have,

(8)
$$\frac{dy}{dx} = \cos u \frac{du}{dx}.$$

Derivative of cos u. From trigonometry

$$\cos u = \sin\left(\frac{\pi}{2} - u\right).$$

By the use of formula (8),

$$\frac{d}{dx}\cos u = \frac{d}{dx}\sin\left(\frac{\pi}{2} - u\right),$$

$$= \cos\left(\frac{\pi}{2} - u\right)\frac{d}{dx}\left(\frac{\pi}{2} - u\right),$$

(9) $$\frac{d}{dx}\cos u = -\sin u\frac{du}{dx}.$$

Derivatives of tan u and cot u. Expressing $\tan u$ in terms of $\sin u$, and $\cos u$,

$$\tan u = \frac{\sin u}{\cos u}.$$

Differentiating the quotient of two functions by formula (6) of Section 35,

$$\frac{d}{dx}\tan u = \frac{\cos u\dfrac{d}{dx}\sin u - \sin u\dfrac{d}{dx}\cos u}{\cos^2 u},$$

$$= \frac{(\cos^2 u + \sin^2 u)\dfrac{du}{dx}}{\cos^2 u}$$

(10) $$\frac{d}{dx}\tan u = \sec^2 u\frac{du}{dx}.$$

In the same manner, $\cot u = \dfrac{\cos u}{\sin u}$,

$$\frac{d}{dx}\cot u = \frac{-(\sin^2 u + \cos^2 u)\dfrac{du}{dx}}{\sin^2 u},$$

(11) $$\frac{d}{dx}\cot u = -\csc^2 u\frac{du}{dx}.$$

Derivatives of sec u and csc u. Expressing $\sec u$ in terms of $\cos u$,

$$\sec u = \frac{1}{\cos u} = (\cos u)^{-1}.$$

Applying formula (7) of Section 35,

$$\frac{d}{dx} \sec u = -(\cos u)^{-2} \frac{d}{dx} \cos u,$$

$$= \frac{\sin u}{\cos^2 u} \frac{du}{dx},$$

(12) $$\frac{d}{dx} \sec u = \sec u \tan u \frac{du}{dx}.$$

In the same manner,

$$\csc u = \frac{1}{\sin u},$$

$$\frac{d}{dx} \csc u = -\frac{\cos u}{\sin^2 u} \frac{du}{dx}$$

(13) $$\frac{d}{dx} \csc u = -\csc u \cot u \frac{du}{dx}.$$

To illustrate the use of the formulas derived, we shall differentiate th
function

$$y = \sin^3 2x \cos 2x.$$

Using differentiation formulas (5), (7), (8), and (9),

$$\frac{dy}{dx} = \sin^3 2x \frac{d}{dx} \cos 2x + \cos 2x \frac{d}{dx} \sin^3 2x,$$

$$= -\sin^4 2x \frac{d}{dx} 2x + 3 \sin^2 2x \cos^2 2x \frac{d}{dx} 2x,$$

$$= -2 \sin^4 2x + 6 \sin^2 2x \cos^2 2x,$$

$$= 6 \sin^2 2x - 8 \sin^4 2x.$$

As a second illustration, let us differentiate

$$y = \frac{\tan^2 x}{\tan x + \sec x}.$$

Applying formulas (6), (10) and (12),

$$\frac{dy}{dx} = \frac{2(\tan x + \sec x) \tan x \sec^2 x - \tan^2 x(\sec^2 x + \sec x \tan x)}{(\tan x + \sec x)^2},$$

$$\frac{dy}{dx} = \frac{\tan x \sec x \,(2 \sec x - \tan x)}{\sec x + \tan x}.$$

Exercise 38

GROUP A

Differentiate each of the following functions.

1. $y = 6 \sin 3x.$

2. $y = 4 \tan \dfrac{x}{2}.$

3. $y = 3 \cos (1 - 2x).$

4. $y = \sin^2 3x.$

5. $y = \sin^2 x \cos^3 x.$

6. $y = \sec 2x \tan 2x.$

7. $y = \cot^3 4x.$

8. $y = 4 \sin^2 \tfrac{1}{4}x + 2x \cos \tfrac{1}{4}x.$

Find the slopes of each of the following curves at the specified points.

9. $y = \sin x$ at $x = \dfrac{\pi}{3}, \dfrac{2}{3}\pi, \pi.$

10. $y = \cos x$ at $x = \dfrac{\pi}{2}, \dfrac{5}{6}\pi, \pi.$

11. $y = \tan 2x$ at $x = -\dfrac{\pi}{2}, \dfrac{\pi}{2}, \dfrac{\pi}{6}.$

12. $y = \sec x$ at $x = 0, \dfrac{\pi}{3}, \dfrac{4}{3}\pi.$

13. $y = \sin x + \cos x$ at $x = 0, \dfrac{\pi}{4}, \dfrac{\pi}{2}, \dfrac{3}{4}\pi, \pi.$

GROUP B

Differentiate each of the following functions.

14. $y = \dfrac{3}{4} \tan^4 \dfrac{x}{3} + \tan^3 \dfrac{x}{3}.$

15. $y = (1 - \sin 2x)^2.$

16. $y = (1 - 2 \tan^2 2x)^3.$

17. $y = \sqrt{1 - \cos x}.$

18. $y = \dfrac{1 + \sin x}{1 - \sin x}.$

19. $y = \dfrac{\sin x}{x}.$

20. $y = \dfrac{\tan^2 3x}{2x}.$

21. $y = \dfrac{\sec x - \tan x}{\sec x + \tan x}.$

22. Derive the formula for the differentiation of $y = \cos u$, where u is a function of x which can be differentiated, using the delta process.

23. If $f(x) = \sin x$, find $f'(x)$, $f''(x)$, $f'''(x)$ and $f^{\mathrm{iv}}(x).$

24. If $f(x) = \tan x$, find $f'(x)$, $f''(x)$ and $f'''(x).$

Find $\dfrac{dy}{dx}$ for each of the following equations.

25. $\sin x + \cos y = 0.$

26. $\cos 2x + \sec 3y = 0.$

27. $\tan (x + y) + \tan (x - y) = 1.$

28. $xy - \tan xy = 0.$

29. $\sin \dfrac{x}{y} + \cos \dfrac{y}{x} = 0.$

30. $y = x \sec \dfrac{y}{x}.$

Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ for each of the following.

31. $x = 4 \cos \theta, \ y = 4 \sin \theta.$

32. $x = a \cos \theta, \ y = b \sin \theta.$

33. $x = a \sec \theta, \ y = b \tan \theta.$

34. Consider the function $y = 1 + \sin 2x$ in the interval $0 \leq x \leq \pi$. Find those intervals in which the function is increasing and is decreasing and those intervals in which the curve is concave upward and is concave downward.

35. Consider the parametric equations of a projectile given in Section 42, where v_0 is constant and the angle θ is variable. Find the range of the projectile and find the value of θ giving the maximum range.

61. Applications of Differentiation of Trigonometric Functions.

Thus far in our study of the calculus, the problems have been restricted to those in which the functions involved were algebraic. We may now remove that restriction and solve problems in which the functions are trigonometric. In fact, many of the problems already solved can be more expeditiously treated by the use of trigonometric functions.

In the application of the differentiation of the trigonometric functions, it is essential to remember that the angle must be measured in radians. The same is to be said for the differentiation of the inverse trigonometric functions which is presented in a later section. Each such differentiation is based on the differentiation of $\sin u$, which in turn, is dependent on the limit of the ratio of $\sin \Delta u$ and Δu, as the latter approaches zero, and this is not unity unless the angle is expressed in radian measure.

A problem in which a minimum value is sought, is chosen as the first illustration of the use of trigonometric functions. If two corridors, one of which is 5 ft. wide, intersect at right angles and if a beam 40 ft. long is to be carried horizontally around the corner, we wish to find the minimum width of the second corridor which will just allow the beam to pass, no allowance being made for its width.

Fig. 56

In Figure 56 any position of the beam AC is represented. Let $EC = y$ and $\angle DAB = \angle EBC = \theta$. Then since $AB = 5 \sec \theta$ and $BC = y \csc \theta$,

$$5 \sec \theta + y \csc \theta = 40.$$

Solving for y and differentiating with respect to θ,

$$\frac{dy}{d\theta} = \frac{5}{\csc \theta} (-\sec \theta \tan \theta + 8 \cot \theta - \sec \theta \cot \theta).$$

Placing the derivative equal to zero, reducing and solving,

$$\sec \theta \tan^2 \theta - 8 + \sec \theta = 0$$

$$\sec^3 \theta - 8 = 0, \quad \sec \theta = 2, \quad \theta = \frac{\pi}{3}.$$

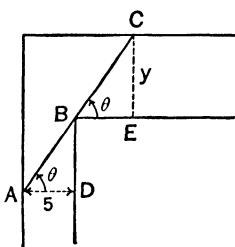Hence, $\qquad AB = 10$ ft., $\quad y = 30 \sin \theta = 15\sqrt{3}$ ft.

If one side of a triangle is 6 ins. long, if a second side is increasing at the rate of 2 ins. per min. and if the included angle is increasing at the rate of 0.8 radians per min., we wish to know how fast the third side is increasing. First, let us assume that the included angle is 60° at the instant the second side is 3 ins. long. Second, let us assume that the second side is 6 ins. long at the instant the included angle is 60°.

Let y represent the length of the second side, z that of the third side and θ the measure of the included angle in radians. By the law of cosines,

$$z^2 = y^2 + 36 - 12y \cos \theta.$$

Differentiating with respect to the time t in minutes,

$$z \frac{dz}{dt} = (y - 6 \cos \theta) \frac{dy}{dt} + 6y \sin \theta \frac{d\theta}{dt}.$$

First, since $z = 3\sqrt{3}$ ins., for $y = 3$ ins. and $\theta = \frac{\pi}{3}$ radians,

$$\frac{dz}{dt} = 2.4 \text{ ins. per min.}$$

Second, since $z = 6$ ins., for $y = 6$ ins.

$$\frac{dz}{dt} = 1 + 2.4 \sqrt{3} = 4.16 \text{ ins. per min.}$$

Exercise 39

GROUP A

1. Find the equations of the tangents to the curve $y = \sin 2x + \cos x$ at $x = 0$ and at $x = \pi/6$.
2. Find the equations of the tangents to the curve $y = \tan 2x$ at $x = 0$ and at $x = \pi/8$.
   In the interval $0 < x < \pi$, show for what values of x each of the following functions is increasing and is decreasing.

   3. $y = \sin x$.
   4. $y = \cos 2x$.

   5. $y = \csc x$.
   6. $y = \sec 2x$.

7. Show that $\tan x$ is increasing and that $\cot x$ is decreasing for all values of x.
   Find the maximum and the minimum values of y in each of the following functions in the interval $0 < x < 2\pi$

   8. $y = \sin x + \cos x$.
   9. $y = \sin x + 2 \cos x$.

   10. $y = \sin x + \cos 2x$.
   11. $y = 2 \sin x + \sin 2x$.

12. Find the angle of intersection of the curves $y = \cos x$ and $y = \cos 2x$ between $x = \pi/2$ and $x = \pi$.
13. Find the values of x in the interval $x = 0$ to $x = 2\pi$ for which the curve $y = \sin x - \cos x$ is concave upward and is concave downward.

14. Find the central angle when the area of a circular sector of given perimeter is maximum.

15. The angle θ between the equal sides of an isosceles triangle is increasing at the rate of $\frac{1}{4}$ radian per min. If the equal sides are 8 ins. long, how fast is the area changing when $\theta = 60°$?

GROUP B

16. Differentiate the function $\dfrac{(1 - \sin \theta)^2}{\cos^2 \theta}$ with respect to θ.

17. If $y = \cos^4 x - \sin^4 x$, show that $\dfrac{d^2y}{dx^2} = -4 \cos 2x$.

18. If $y = a \cos k\theta + b \sin k\theta$, show that $\dfrac{d^2y}{d\theta^2} = -k^2 y$.

19. Find the angle of intersection of the curves $y = 2 \sin 2x$ and $y = \tan 2x$ at the common point $0 < x < \pi/4$.

20. If the side a and the opposite angle α of a triangle are given, prove that the maximum triangle is isosceles.

21. Two corridors intersect at right angles, one 27 ft. wide and the other 8 ft. wide. Find the length of the longest beam which can be carried horizontally around the corner, neglecting the width of the beam.

22. Find the vertical angle of a conical vessel of minimum volume which will permit a sphere of radius a to be completely submerged if the vessel is filled with water.

23. At any time t a moving particle has the position given by the equations $x = 2 - 4 \cos t$, $y = 3 + 2 \sin t$. Find the positions and the velocities for $t = 0$ and for $t = \pi/2$. Show that at one time the velocity is maximum and at the other it is minimum. Draw the curve of the path.

24. The position of a particle in the xy-plane is given by the equations $x = 2(\theta - \sin \theta)$, $y = 2(1 - \cos \theta)$. If the angle θ is increasing at the rate of 2 radians per min., find the velocity and the acceleration of the particle when $\theta = \pi/3$.

25. Approximate the value of $\sin 60° \, 10'$ and find the approximate relative error.

26. Find the approximate maximum relative error in the cosine of an angle if the angle is measured as $\pi/6$ with a possible maximum error of 0.01 radian.

62. Simple Harmonic Motion.

In the study of physics there are important applications of the motion of a particle which is vibrating in a straight line. This motion is typified by that of a particle supported by a vertical spiral spring and vibrating freely in a vertical line. Another illustration is furnished by the motion of any point on a guitar string which has been plucked and released. Other important examples are to be found in the motion of a particle transmitting a wave of sound, a wave of light or an electric impulse. In these illustrations there are forces acting which tend to bring the particle to rest after a period of time. In the discussion which follows, we shall neglect such forces, thus simplifying the problem and bringing it within the scope of our treatment.

Suppose that a particle P moves on a straight line and that its distance from a fixed point O of that line at any time t is denoted by the directed variable s. Also suppose that the particle starts motion at the point O, then $s = 0$ when $t = 0$. Since the sine of an angle is a periodic function, we write

$$s = a \sin bt,$$

and study the motion of the particle under such circumstances, where a and b are constants. We note the positions of the particle at each of the following instants of time:

If $t = n\pi/2b$,

$$n = 0, 2, 4, 6, \cdots, \qquad s = 0 \text{ and it is at the point } O.$$

But for

$$n = 1, 5, 9, 13, \cdots, \qquad s = a \text{ and it is at the point } A.$$

And for

$$n = 3, 7, 11, 15, \cdots, \qquad s = -a \text{ and it is at the point } A'.$$

See Figure 57. Consequently, we have written an expression for the distance of a particle from a fixed point as a function of the time in which the particle vibrates between the points A a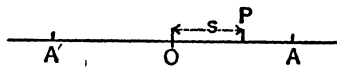nd A', starting its motion at the *mean position* O. If the constant a is positive, the particle starts motion to the right, but if a is negative, it starts to the left. The constant b we shall always assume positive.

Fig. 57

The absolute value of the constant a is called the *amplitude* of the motion. When $t = 2\pi/b$, the particle has completed one complete oscillation. This interval of time is called the *period* of the motion.

Since the cosine of an angle is also a periodic function, we might choose to write

$$s = a \cos bt.$$

An investigation will show that this function also expresses the distance of a particle from the point O at any time in which it vibrates between the points A and A', but in which it starts its motion at an *extreme* position, A or A', instead of at the mean position.

Returning to the first equation, the derivative of s with respect to t gives the velocity of the particle at any time.

$$v = ab \cos bt.$$

The differentiation of v with respect to t, gives the acceleration of the particle at any time,

$$j = -ab^2 \sin bt.$$

If we form the ratio of the acceleration j to the displacement s,

$$\frac{j}{s} = -\frac{ab^2 \sin bt}{a \sin bt} = -b^2,$$

we find that it is independent of the time and is equal to the constant $-b^2$. In other words, the motion is one in which the acceleration is proportional to the displacement and opposite in sense. Such a motion is called *simple harmonic motion*.

Let us analyze the motion of a particle for which it is given that

$$s = 8 \cos 3t.$$

When $t = 0$, $s = 8$ and the motion starts at the point A in Figure 57. When $t = \pi/6$, $s = 0$ and the particle has reached the mean position O. When $t = \pi/3$, $s = -8$ and it has reached the extreme position A'. When $t = \frac{2}{3}\pi$, it has returned to the extreme position A. The amplitude of the motion is 8 and the period is $\frac{2}{3}\pi$. Since

$$v = -24 \sin 3t \quad \text{and} \quad j = -72 \cos 3t, \quad \frac{j}{s} = -9.$$

Hence, the motion is simple harmonic.

It is not always possible to find the period and the amplitude of a simple harmonic motion by inspection. For example, in the function

$$s = 3 \cos 4t - 4 \sin 4t,$$

we proceed as follows:

$$v = -12 \sin 4t - 16 \cos 4t.$$

$$j = -48 \cos 4t + 64 \sin 4t.$$

When $t = 0$, $s = 3$ and $v = -16$, thus giving the initial position and the direction of motion at the instant of starting. To find the extreme positions, we find the maximum and minimum values of s. For $v = 0$,

$$\tan 4t = -\tfrac{4}{3}, \quad \sin 4t = \tfrac{4}{5}, \quad \cos 4t = -\tfrac{3}{5},$$

giving a positive value of j. From these values $s = -5$, which is the minimum value of s. But also for $v = 0$,

$$\tan 4t = -\tfrac{4}{3}, \quad \sin 4t = -\tfrac{4}{5}, \quad \cos 4t = \tfrac{3}{5},$$

giving a negative value of j. From these values $s = 5$, which is the maximum value of s. Hence, the amplitude is 5 and the fixed point is the mean position. Moreover, when

$$t = \frac{\pi}{2}, \quad s = 3, \quad v = -16.$$

At the instant $t = \pi/2$, the particle passes the initial position moving in the same direction. Hence, $\pi/2$ is the period of the motion. The motion is simple harmonic, since

$$\frac{j}{s} = -16.$$

Exercise 40

GROUP A

1. Analyze the motion of a particle moving on a line such that its distance from a fixed point is $s = 4 \sin 2t$.
2. As in Problem 1, analyze the motion given by $s = -10 \cos 3t$.
3. A particle moves with simple harmonic motion having a period of 8 secs. and an amplitude of 5 ft. If it starts at the mean position moving to the left, find the equation and the velocity at the second passing of the mean position.
4. A particle moves with simple harmonic motion having a period of 4 secs. and an amplitude of 6 ft. If it starts motion at the left extreme, find the equation and the acceleration at the right extreme.
5. Given $s = -12 \sin \frac{3}{4}t$. Find the velocity and acceleration when $t = \pi/3, \pi/2, 2\pi/3$, and 2π secs.
6. Analyze the motion of a particle whose position is given by $s = 5 + 3 \sin \frac{t}{2}$.
   Show that the displacement from the mean is proportional to the acceleration.
7. A particle starts at its mean position and has a period of 8π secs. Find the equation if $s = 2$ at the end of $2\pi/3$ secs.
8. A particle starts at its right hand extreme and has a period of 3π secs. If $s = 3$ when $v = 2$, find the equation and its velocity as it passes its mean position.

GROUP B

9. Show that the function $s = 2 - 4 \sin^2 2t$ gives the simple harmonic motion of a particle. Find its period and amplitude.
10. Show that the function $s = 4 - 8 \cos^2 3t$ gives the simple harmonic motion of a particle. Find its period and amplitude.
11. Show that the function $s = 4 \sin \frac{t}{3} - 3 \cos \frac{t}{3}$ gives the simple harmonic motion of a particle. Find its velocity and acceleration at the mean position.
12. A particle moves with simple harmonic motion with an amplitude of 3 ft. If its velocity is 5 ft. per sec. when at a distance of 2 ft. from its mean position, find its period.

13. A particle moves with simple harmonic motion with a period of 4 secs. If it has a velocity of 2π ft. per sec. when it is at a distance of 3 ft. from its mean position, find its amplitude.

14. A particle moves with simple harmonic motion with an amplitude of 4 ft. If its velocity is 4 ft per sec. when it is halfway from its mean to its extreme position, find the period of the motion.

15. Find the volume generated by the rotation of a parabolic segment, base b and altitude a, about its base.

16. Find the volume cut from a right circular cylinder, radius of base r and altitude a, by a plane through the diameter of the base and tangent to the upper base.

63. Graphs of Inverse Trigonometric Functions.

The statement, u is the sine of the angle y, is written

$$u = \sin y.$$

The inverse statement, y is the angle whose sine is u, is written

$$y = \arcsin u.$$

The symbol arcsin u is called the *inverse sine function of u*. It is represented by the angle y, where it is to be understood that y is measured in radians. Moreover, the function is defined for limited values of u, that is, for

$$-1 \leqq u \leqq 1.$$

There are six elementary inverse trigonometric functions,

arcsin u, arccos u, arctan u, arccot u, arcsec u, arccsc u,

where each is defined by means of the corresponding trigonometric function. Each inverse function is a multi-valued function, that is, for every value of u, within its limits, there are many values of the angle y.

Principal Values. For most purposes of the calculus where inverse functions are used, it is advisable, and often necessary, to limit the range of such functions. A *principal value of a function* is one which lies within a range of values which is selected for each function. If the principal values only, are used, we shall have under consideration a single-valued function. The principal values of

$$\arcsin u, \arctan u \text{ and arccsc } u \text{ are } -\frac{\pi}{2} \leqq u \leqq \frac{\pi}{2}$$

and of

$$\arccos u, \operatorname{arccot} u \text{ and arcsec } u \text{ are } 0 \leqq u \leqq \pi.$$

Graph of $y = a$ arcsin bx. In drawing the graph of the function

$$y = 2 \text{ arcsin } 3x,$$

we observe that x is limited to values from $x = -\frac{1}{3}$ to $x = \frac{1}{3}$.

If $x = 0$, $\quad y = \cdots, -4\pi, -2\pi, 0, 2\pi, 4\pi, \cdots$.

If $x = -\frac{1}{3}, y = \cdots, -5\pi, -\pi, 3\pi, 7\pi, \cdots$.

If $x = \frac{1}{3}$, $\quad y = \cdots, -7\pi, -3\pi, \pi, 5\pi, \cdots$.

The curve is drawn in Figure 58 from $y = -4\pi$ to $y = 4\pi$. That part of the curve corresponding to the principal values of the function, from $y = -\pi$ to $y = \pi$, is more heavily drawn than are the other parts.
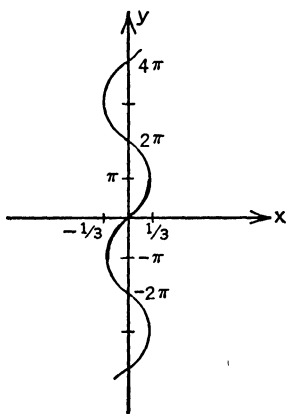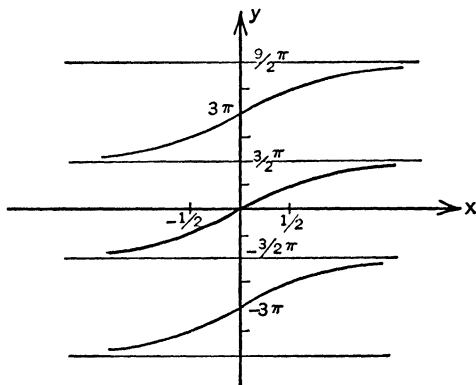

Fig. 58

Fig. 59

Graph of $y = a$ arctan bx. In drawing the graph of the function

$$y = 3 \text{ arctan } 2x,$$

it is observed that the values of the variable $2x$ are unlimited. If

$$x = 0, y = \cdots, -6\pi, -3\pi, 0, 3\pi, 6\pi, \cdots.$$

As x increases or decreases without limit from zero, the corresponding values of y locate the positions of the horizontal asymptotes. Thus, if

$$x = \infty, y = \cdots, -\tfrac{9}{2}\pi, -\tfrac{3}{2}\pi, \tfrac{3}{2}\pi, \tfrac{9}{2}\pi, \cdots.$$

Three branches of the curve are drawn in Figure 59 in which the central branch corresponds to the principal values of the function.

Exercise 41

Solve for x in each of the following equations and reduce to the simplest form.

1. $y = 2 \arccos 3x$.
2. $y = \frac{2}{3} \arcsin \frac{3}{2}x$.
3. $y = \pi - \arcsin x$.
4. $y = 2 \arccos (x - 1) - \pi$.

Draw the graphs of each of the following functions.

5. $y = \arcsin \dfrac{x}{2}$
6. $y = 2 \arccos x$.
7. $y = \arctan 2x$.
8. $y = \frac{1}{2} \arcsin 2x$.
9. $y = 3 \arccos \dfrac{x}{2}$
10. $y = 3 \operatorname{arcsec} 2x$.

GROUP B

Draw the graph of each of the following functions.

11. $y = \frac{1}{3} \arcsin (2x - 3)$.
12. $y = \frac{1}{2} \arccos (2x - 1)$.
13. $y + \pi/2 = \arcsin (2x + 1)$.
14. $y = \pi/2 - \arcsin (x + 2)$.

In each of the following solve for x and differentiate implicitly. Solve for dy/dx and express the result as a function of x.

15. $y = \arcsin 2x$.
16. $y = 2 \arccos 3x$.
17. $y = \arctan \dfrac{x}{2}$
18. $y + 2 = \operatorname{arcsec} x$.
19. $y = \pi - \operatorname{arccsc} 2x$.
20. $y = \pi/2 - \operatorname{arccot} 3x$.

Verify each of the following identities.

21. $\arctan \frac{1}{3} - \arctan \frac{1}{13} = \arctan \frac{1}{4}$.
22. $\arcsin \frac{5}{13} + \arcsin \frac{12}{13} = \pi/2$.
23. $\cos (2 \arctan x) = \dfrac{1 - x^2}{1 + x^2}$.
24. $2 \arctan 2 + \arctan \frac{4}{3} = \pi$.
25. $\sin (2 \arccos x) = 2x\sqrt{1 - x^2}$.

64. Formulas for Differentiation of Inverse Trigonometric Functions.

The formulas for the differentiation of the six inverse trigonometric functions are as follows, where u denotes any function of x which can be differentiated:

$$(14) \qquad \frac{d}{dx} \arcsin u = \frac{1}{\sqrt{1 - u^2}} \frac{du}{dx}.$$

$$(15) \qquad \frac{d}{dx} \arccos u = \frac{-1}{\sqrt{1 - u^2}} \frac{du}{dx}.$$

$$(16) \qquad \frac{d}{dx} \arctan u = \frac{1}{1 + u^2} \frac{du}{dx}.$$

(17) $$\frac{d}{dx} \text{arccot } u = \frac{-1}{1 + u^2} \frac{du}{dx}.$$

(18) $$\frac{d}{dx} \text{arcsec } u = \frac{1}{u\sqrt{u^2 - 1}} \frac{du}{dx}.$$

(19) $$\frac{d}{dx} \text{arccsc } u = \frac{-1}{u\sqrt{u^2 - 1}} \frac{du}{dx}.$$

In each of these formulas the inverse function is understood to be restricted to principal values.

65. Derivation of Inverse Trigonometric Differentiation Formulas.

To derive the formula for the differentiation of arcsin u, let

$$y = \text{arcsin } u,$$

where u is any function of x which can be differentiated. Then

$$u = \sin y.$$

Differentiating this equation with respect to x,

$$\frac{du}{dx} = \cos y \frac{dy}{dx},$$

whence, $$\frac{dy}{dx} = \frac{1}{\cos y} \frac{du}{dx}.$$

But since $$\cos y = \pm\sqrt{1 - \sin^2 y} = \pm\sqrt{1 - u^2},$$

$$\frac{dy}{dx} = \frac{\pm 1}{\sqrt{1 - u^2}} \frac{du}{dx}.$$

If the angle arcsin u is restricted to the first and fourth quadrant angles, that is,

$$-\frac{\pi}{2} \leqq \text{arcsin } u \leqq \frac{\pi}{2},$$

the principal values of the function are retained. A reference to Figure 58, Section 63, will show that the slope of the limited portion of the curve corresponding to the principal values of the function is everywhere positive. Consequently, the negative sign in the derivative is discarded and we write

(14) $$\frac{d}{dx} \text{arcsin } u = \frac{+1}{\sqrt{1 - u^2}} \frac{du}{dx},$$

which is understood to be the derivative of the function for principal values of the function only.

To derive the formula for the differentiation of arccos u, let

$$y = \arccos u.$$

Then $\qquad u = \cos y$

and $\qquad \dfrac{dy}{dx} = \dfrac{-1}{\sin y}\dfrac{du}{dx} = \dfrac{\mp 1}{\sqrt{1 - u^2}}\dfrac{du}{dx}.$

If the angle arccos u is restricted to the principal values, the slope of the part of the curve under consideration is everywhere negative. Consequently, the positive sign is discarded and we write

$$(15) \qquad \frac{d}{dx}\arccos u = \frac{-1}{\sqrt{1 - u^2}}\frac{du}{dx}.$$

To derive the formula for the differentiation of arctan u, let

$$y = \arctan u.$$

Then $\qquad u = \tan y$

and $\qquad \dfrac{dy}{dx} = \dfrac{1}{\sec^2 y}\dfrac{du}{dx} = \dfrac{1}{1 + u^2}\dfrac{du}{dx},$

which is formula (16).

The derivatives of arccot u, arcsec u and arccsc u are obtained in a like manner, obtaining formulas (17), (18) and (19), respectively. These derivations are left as an exercise.

Exercise 42

GROUP A

Differentiate each of the following functions.

1. $y = 2\arcsin 3x.$

2. $y = \arccos \frac{3}{2}x.$

3. $y = \arctan \dfrac{1 - x}{2}.$

4. $y = x \arctan x.$

5. $y = \arcsin x^2.$

6. $y = \arcsin \frac{2}{3}\sqrt{x}.$

7. $y = \arcsin \sqrt{x + 1}.$

8. $y = \arcsec(x^2 + 1).$

9. $y = x\,\text{arccot}\,\dfrac{x}{2} + \dfrac{1}{x^2 + 4}.$

•10. $y = x \arccsc x + \sqrt{x^2 - 1}.$

11. Find the principal maximum and minimum values of the function

$$y = x - 4\arctan\frac{x}{2}.$$

12. One side of a right triangle is increasing at the rate of 2 ins. per sec., while the second side remains 8 ft. long. At what rates are the hypotenuse and the angle opposite the second side changing when the first side is 6 ft. long?

GROUP B

Differentiate each of the following functions.

13. $y = 9 \arcsin \dfrac{x}{3} - x \sqrt{9 - x^2}$.

14. $y = \arcsin \dfrac{x}{2} + \dfrac{\sqrt{4 - x^2}}{x}$.

15. $y = x \arcsin x + \sqrt{1 - x^2}$.

16. $y = \dfrac{x}{\sqrt{16 - x^2}} - \arcsin \dfrac{x}{4}$.

17. $y = (x^2 + 1) \arctan x - x$.

18. $y = x \arctan x^2$.

19. $y = a \arcsin \dfrac{x}{a}$.

20. $y = \dfrac{1}{a} \arctan \dfrac{x}{a}$.

21. $y = \arctan \dfrac{a}{x}$.

22. $y = \sqrt{x^2 - a^2} + a \arcsin \dfrac{a}{x}$.

Derive the formula for the derivative of each of the following functions, assuming that u is a function of x which can be differentiated.

23. arccot u.   **24.** arcsec u.   **25.** arccsc u.

26. Find $\dfrac{dy}{dx}$ from $y = \arcsin (xy)$.

27. Find $\dfrac{dy}{dx}$ from $y = \arctan \dfrac{x}{y}$.

28. Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ from $x = \arctan (xy)$.

29. Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ from $x = \arctan \dfrac{y}{x}$.

30. Find $\dfrac{d\theta}{dx}$ and $\dfrac{d^2\theta}{dx^2}$ from $\tan \theta = \dfrac{x + 1}{x}$.

31. Find the principal maximum and minimum values of the function $y = (x - 1) - 4 \arctan (x - 1)/2$. Find the coordinates of the inflection point of the curve and give the values of x for which the curve is concave upward and downward.

32. The altitude of a right circular cone is 3 ft. If the radius of the base increases at the rate of 2 ins. per sec., find the rate of change of the vertical angle of the cone.

33. A pole 15 ft. long is resting against a vertical wall. If the foot of the pole is pulled away from the wall at the rate of 2 ft. per sec., how fast is the angle with the ground decreasing when the foot is 12 ft. from the wall?

34. A searchlight is 10 miles from a straight railroad track. If a train travelling along the track at the rate of 60 mi. per hr. is followed by the light, how fast is the light rotating when the train is 20 miles down the track from the point nearest the light?

35. Find the force exerted on a vertical water gate which has the shape of a parabolic segment, base b and altitude a, where the base is c feet below the surface of the water, the vertex of the parabola being below the base.

66. Angular Velocity and Acceleration.

If a wheel rotates on its axis, a spoke of the wheel generates an angle which is dependent on the time of rotation. Denoting such an angle by θ, it is a function of the time,

$$\theta = f(t).$$

The rate of change of θ with respect to the time is called the *angular velocity* and shall be represented by ω. Hence,

$$\omega = \frac{d\theta}{dt} = f'(t).$$

The rate of change of the angular velocity with respect to the time is called the *angular acceleration* and shall be represented by α. Hence,

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} = f''(t).$$

If θ is measured in radians and t in seconds, the angular velocity is in radians per second and the acceleration is in radians per second per second. Angular velocity divided by 2π radians gives the number of revolutions of the wheel per second, if ω is constant.

As a point $P(x,y)$ on the rim of a wheel of radius a describes an arc of length s, the radius sweeps over an angle θ at the center. From Section 56,

$$s = a\theta.$$

Differentiating with respect to t,

$$\frac{ds}{dt} = a\frac{d\theta}{dt}, \quad \text{or} \quad v = a\omega,$$

where v is the velocity of the point P along the circle. Thus for circular motion, the linear velocity is proportional to the angular velocity, the proportionality constant being the radius.

Suppose that a wheel of radius 2 ft. makes 3 revolutions per second and that we wish to find the horizontal and vertical components of the velocity of a point on the rim 1 ft. above the level of the center.

Taking the origin at the center of the wheel and the x-axis horizontal,

$$x = 2\cos\theta, \quad y = 2\sin\theta.$$

Differentiating with respect to t,

$$v_x = -2\omega\sin\theta = -12\pi\sin\theta$$

$$v_y = 2\omega\cos\theta = 12\pi\cos\theta,$$

since $\omega = 6\pi$. For $y = 1$, $\theta = \pi/6$ or $\theta = 5\pi/6$. Hence, for the first,

$$v_x = -6\pi, \quad v_y = 6\sqrt{3}\,\pi,$$

which is verified by $|v| = 12\pi$. For the second value of the angle,

$$v_x = -6\pi, v_y = -6\sqrt{3}\,\pi.$$

In the same problem,

$$\dot{j}_x = -72\pi^2 \cos\theta, \quad \dot{j}_y = -72\pi^2 \sin\theta,$$

$$|\dot{j}| = 72\pi^2 \quad \text{and} \quad \tan\phi = \frac{\dot{j}_y}{\dot{j}_x} = \tan\theta.$$

From the last statement, the linear acceleration of the point on the rim is directed toward the center of the wheel. The angular acceleration is zero, since the angular velocity is constant.

Again in the same problem, let us ask what constant acceleration would bring the wheel from rest to its present angular velocity in 30 seconds?

Since $$\frac{d\omega}{dt} = \alpha, \quad \omega = \int \alpha dt,$$

and $$\omega = \alpha t + C.$$

But $\omega = 0$ when $t = 0$, hence $C = 0$ and $\omega = \alpha t$.

When $t = 30$ secs., $\alpha = \pi/5$ rads. per sec.^{2}

To consider a problem of a more general nature, suppose that a point $P(x,y)$ describes a locus in the xy-plane and that, under certain conditions, it is required to find the angular velocity of the line determined by P and a fixed point O.

Let the fixed point O be the origin, then

$$\tan\theta = \frac{y}{x}, \quad \theta = \arctan\frac{y}{x}.$$

Differentiating with respect to t,

$$\omega = \frac{xv_y - yv_x}{x^2 + y^2}.$$

Exercise 43

GROUP A

1. A point on the rim of a wheel of radius 5 ft. has the vertical component of its velocity 50 ft. per min. when it is 4 ft. above the level of the center. Find the angular velocity.

2. A point on the rim of a wheel of radius 10 ft. has the horizontal component of its velocity 100 ft. per min. when it is 6 ft. above the level of the center. Find the number of revolutions of the wheel per min.

3. The angular velocity of a point on the rim of a wheel is $6t$ ft. per second. Find the angle passed over by a radius in the first two seconds and in the first four seconds.

4. The angular acceleration of a point on the rim of a wheel is $12t$ ft. per sec. per sec. If the wheel starts rotation from rest, find the angle passed over by a radius from $t = 1$ sec. to $t = 3$ secs.

5. The position of a moving point at any time is given by $x = 5 \cos 2t$, $y = 5 \sin 2t$. Show that the point moves on a circle and find the angular velocity of a radius.

6. The position of a moving point at any time is given by $x = 4 \cos 3t$, $y = 4 \sin 3t$. Find the angular velocity of a radius and the angle passed over during the first three seconds.

7. Find the angular velocity of the line through the origin and $P(x,y)$ at the point $(2,1)$ when x and y each increase at the rate of 4 ins. per sec.

8. A point moves on a circle of radius 3 ft. so that its distance along the arc from a fixed point on the circle is given by $s = 9t^2$. Find the angular velocity at the end of 2 secs. and find the angular acceleration at any time.

GROUP B

9. The position of a moving point at any time is given by $x = 3 \cos t$, $y = 2 \sin t$. Find the equation of the path and linear velocity and position when $t = 0$ and when $t = \pi/4$.

10. One side of a right triangle is 6 ft. long and the adjacent angle is increasing at the rate of 3 radians per min. Find the rate of increase of the other side when it is 2 ft. long.

11. A lighthouse at A is due east 6 mi. from a point B at sea. A ship moving north from B is followed by a beam of light from A. If the ship is moving 12 mi. per hr., how fast is the beam of light rotating when the ship is 8 mi. north of B?

12. A point is moving on the line $3x + 4y = 27$ in such a way that $x = 4t^2$. Find the angular velocity of the line joining the point to the origin when $t = \frac{1}{2}$.

13. A point moves on the line $x + 2y = 8$ with a linear velocity of 6 ft. per sec. Find the angular velocity of the line joining the point to the origin as it passes through the point $(2,3)$.

14. Find the angular velocity of the line $A(a,0)$ $P(x,y)$ if x and y each increase at the rate of b units per sec. as the point P reaches the position $B(3a,a)$.

15. A point P moves on the curve $x^2 - y^2 = 16$ so that the y-component of its velocity is 5 ins. per sec. Find the velocity at the point $A(5,3)$. Find the rate of change of the angle which OP makes with the x-axis as P reaches the position at A.

16. A point moves on the curve $y = 4 \cos (2x - 3)$ so that the x-component of the velocity is 2 ins. per sec. Find the coordinates of two points, one at which the velocity is maximum and one, at which it is minimum.

67. The Cycloid.

A *cycloid* is a curve traced by a point on the circumference of a circle as it rolls along a straight line.

To derive the parametric equations of the cycloid, let a circle of radius a roll on the x-axis. Also let the point $P(x,y)$ which describes the locus be that point on the circumference which is the point of contact when the circle is tangent to the x-axis at the origin. If the circle rolls from left to right, the radius CP in Figure 60 has turned through the angle with the vertical,

$$\theta = \angle NCP,$$
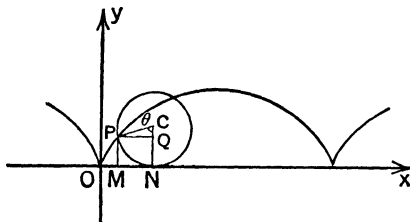
Fig. 60

since the time P occupied the position at the origin. The arc NP is equal in length to the line segment ON, because we assume that there is no slipping as the circle rolls. From the figure

$$x = ON - PQ, \quad y = NC - QC,$$

$$ON = a\theta, \quad PQ = a \sin \theta, \quad QC = a \cos \theta.$$

Hence, $$x = a(\theta - \sin \theta), \quad y = a(1 - \cos \theta).$$

The x- and the y-components of the velocity of a point on the rim of a rolling wheel are dependent on the angular rotation of the wheel. Thus

$$v_x = a\omega(1 - \cos \theta), \quad v_y = a\omega \sin \theta$$

and $$v = a\omega\sqrt{2 - 2 \cos \theta} = 2 a\omega \sin \frac{\theta}{2}.$$

From the components of the acceleration, assuming ω constant,

$$j_x = a\omega^2 \sin \theta, \quad j_y = a\omega^2 \cos \theta,$$

we have $$j = a\omega^2.$$

Exercise 44

GROUP A

1. A wheel of radius 2 ft. rolls on the level ground making 10 revolutions per sec. Find the linear velocity of a point on the rim when the radius through the point makes 60° with the vertical.

2. Find the slope of the cycloid for $\theta = 60°$ and for $\theta = \frac{2}{3}\pi$.

3. Find the components of the velocity of a point describing a cycloid when it reaches the highest point and the lowest point.

4. Find the position of a point moving on a cycloid when the x-component of the velocity is maximum.

5. Find the position of a point moving on a cycloid when the y-component of the velocity is a maximum.

6. Show that the x-component of the velocity of a point moving on a cycloid is proportional to the ordinate of the point.

7. Prove that the slope of the cycloid at any point is $\cot \dfrac{\theta}{2}$.

8. Find the linear velocity of a point moving on a cycloid when it reaches a position whose ordinate is half the radius of the rolling circle.

GROUP B

9. Find the rectangular equation of the cycloid.

10. If the equations of a cycloid are $x = 3(2t - \sin 2t)$, $y = 3(1 - \cos 2t)$, find the positions and velocities of a point on the generating circle when $t = \pi/4$ and $t = \pi/2$.

11. If a circle with the radius of 6 ins. rolls on the x-axis so that the angular velocity of a radius CP is 3 radians per sec., find, in terms of the parameter t, the equations of the cycloid traced by the point P starting at the origin.

12. Using the same conditions as given in Problem 11, find the equations of the cycloid traced by the point P starting so that $\theta = \pi/2$ when $t = 0$.

13. The parametric equations of a curve, called a *trochoid*, traced by a point $P(x,y)$ on the radius, or the radius produced, of a circle of radius a are $x = a\theta - b\sin\theta$, $y = a - b\cos\theta$, where $b = CP$. Find the components of the velocity and the velocity of the point P.

14. Find the components of the acceleration and the acceleration of the point P moving on the trochoid given in Problem 13, where the angular velocity of CP is constant.

68. Integration.

It was stated in Chapter IV that an indefinite integral of a function is a function whose derivative is the given function. It was also pointed out that while differentiation is a direct process, integration is indirect and is performed by reversing the result of a differentiation.

With these statements in mind, the formulas for the integration of certain trigonometric functions and special algebraic functions can be written immediately from the differentiation formulas derived in Sections 60 and 65 of this chapter. They are as follows:

(4) $$\int \sin u \; du = -\cos u + C.$$

(5) $$\int \cos u \; du = \sin u + C.$$

(6)
$$\int \sec^2 u \; du = \tan u + C.$$

(7)
$$\int \csc^2 u \; du = -\cot u + C.$$

(8)
$$\int \tan u \sec u \; du = \sec u + C.$$

(9)
$$\int \cot u \csc u \; du = -\csc u + C.$$

(10)
$$\int \frac{du}{\sqrt{1 - u^2}} = \arcsin u + C.$$

(11)
$$\int \frac{du}{1 + u^2} = \arctan u + C.$$

(12)
$$\int \frac{du}{u\sqrt{u^2 - 1}} = \operatorname{arcsec} u + C.$$

In these formulas it is understood that u is a function of some variable thus making du its differential.

The following integrations are carried out as illustrations of the formulas:

$$\int \sin 6x \; dx = \tfrac{1}{6} \int \sin 6x(6 \; dx) = -\tfrac{1}{6} \cos 6x + C,$$

since $u = 6x$ and $du = 6 \; dx$.

$$\int (\cos^2 2x - \sin^2 2x) \; dx = \tfrac{1}{4} \int \cos 4x \; (4 \; dx) = \tfrac{1}{4} \sin 4x + C,$$

since $\cos^2 2x - \sin^2 2x = \cos 4x$, $u = 4x$ and $du = 4 \; dx$.

$$\int \tan^2 3x \; dx = \tfrac{1}{3} \int \sec^2 3x(3 \; dx) - \int dx = \tfrac{1}{3} \tan x - x + C,$$

since $1 + \tan^2 3x = \sec^2 3x$.

$$\int \frac{dx}{1 + 2x^2} = \frac{1}{\sqrt{2}} \int \frac{\sqrt{2} \; dx}{1 + 2x^2} = \frac{1}{\sqrt{2}} \arctan \sqrt{2} \, x + C,$$

since $u^2 = 2x^2$, $u = \sqrt{2} \, x$ and $du = \sqrt{2} \; dx$.

The limits of a definite integral of a trigonometric function are expressed in radians. The geometric interpretation of such an integral is the area

under the curve which represents the function. For example, the area under the curve

$$y = \cos \frac{x}{2}$$

from $-\pi/3$, to $\pi/3$, is found as follows:

$$\int_{-\pi/3}^{\pi/3} \cos \frac{x}{2}\, dx = 2 \int_{-\pi/3}^{\pi/3} \cos \frac{x}{2}\, \frac{dx}{2} = 2 \sin \frac{x}{2} \Big]_{-\pi/3}^{\pi/3} = 2\left[\frac{1}{2} - \left(-\frac{1}{2}\right)\right] = 2.$$

In finding the definite integral of a function which gives rise to an inverse trigonometric function, the principal values of that function are used in every instance. For example,

$$\int_{-1/6}^{1/6} \frac{dx}{\sqrt{1 - 9x^2}} = \frac{1}{3} \int_{-1/6}^{1/6} \frac{3\, dx}{\sqrt{1 - 9x^2}} = \frac{1}{3} \arcsin 3x \Big]_{-1/6}^{1/6}$$

$$= \frac{1}{3}\left[\arcsin \frac{1}{2} - \arcsin \left(-\frac{1}{2}\right)\right] = \frac{1}{3}\left[\frac{\pi}{6} - \left(-\frac{\pi}{6}\right)\right] = \frac{\pi}{9}.$$

Exercise 45

GROUP A

Perform each of the following indicated integrations.

1. $\int \sin 3x\, dx.$

2. $\int \cos 4x\, dx.$

3. $\int \sec^2 \frac{x}{2}\, dx.$

4. $\int \sec 2x \tan 2x\, dx.$

5. $\int \frac{2\, dx}{1 + 4x^2}.$

6. $\int \csc^2 \frac{2}{3} x\, dx.$

7. $\int 4 \sin x \cos x\, dx.$

8. $\int \cot^2 x\, dx.$

9. $\int \frac{2\, dx}{\sqrt{1 - 4x^2}}.$

10. $\int \frac{dx}{x\sqrt{x^2 - 1}}.$

11. Find the area under the first arch of the curve $y = \sin x$.

12. Find the area under one arch of the curve $y = \cos x$.

GROUP B

Perform each of the following indicated integrations.

13. $\int \sin 3x \cos 3x\, dx.$

14. $\int \left(\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}\right) dx.$

15. $\int \tan^2 \frac{x}{3}\, dx.$

16. $\int \frac{dx}{\sqrt{1 - 3x^2}}$

17. $\int \sin^2 x \, dx.$

18. $\int \cos^2 x \, dx.$

19. $\int \sin^2 2x \cot 2x \, dx.$

20. $\int \dfrac{dx}{4 + x^2}.$

21. Evaluate $\displaystyle\int_{-3\pi/4}^{3\pi/4} \sec^2 \dfrac{x}{3} \, dx.$

22. Evaluate $\displaystyle\int_{0}^{2\pi/3} \sec \dfrac{x}{2} \tan \dfrac{x}{2} \, dx.$

23. Draw the curve $y = \sin 4x$ and find the area under one arch.

24. Draw the curve $y = \cos \dfrac{x}{2}$ and find the area under one arch.

25. Draw the curve $y(x^2 + 1) = 2$ and find the area under it from $x = -1$ to $x = 1$.

26. Draw the curve $y^2(1 - x^2) = 1$ and find the area under the upper branch between the lines $2x + 1 = 0$ and $2x - 1 = 0$.

27. A particle moves so that the x- and y-components of its velocity at any time are given by $v_x = -2 \sin t$ and $v_y = 2 \cos t$. If the particle has the position $(2,0)$ when $t = 0$, find the parametric equations and the Cartesian equation of its path. Find the linear and angular velocity and the linear acceleration at any time.

28. A particle moves so that the x- and y-components of its velocity at any time are given by $v_x = 2 \sin t$ and $v_y = 3 \cos t$. If the particle has the position $(0,4)$ when $t = 0$, find the parametric equations and the Cartesian equation of its path. Find the linear velocity and acceleration at any time.

29. A particle moves on a circle in such a way that its angular velocity at any time is given by $\omega = \cos \dfrac{t}{2}$. Find the angle passed over by the radius from $t = 0$ to $t = 2\pi$.

30. A particle moves on a line in simple harmonic motion so that its velocity at any time is given by $v = 4 \cos 2t$. If $s = 2$ when $t = 0$, find the amplitude and the period. Find the position and the acceleration when $t = \pi/4$.

GROUP C

31. Find a maximum and a minimum value of $\dfrac{\sin x}{1 + \tan x}.$

32. A picture 5 ft. high hangs vertically on the wall so that its lower edge is 4 ft. above the level of an observer's eye. How far from the wall should the observer stand in order that the picture subtend the greatest angle at his eye?

33. A searchlight 3 miles from a straight level road is following a motor car travelling along the road at the rate of 40 miles per hr. Find how fast the light is rotating as the car passes the point nearest the light and how fast it is rotating 15 mins. later.

34. Two line segments AB and BC intersect at an angle of $60°$. If $AB = 10$ ins., and if a point P is moving from B toward C at the rate of 10 ins. per min., find how fast the line AP is rotating and find how long after P leaves the point B the line AP is rotating most rapidly.

35. A point P moves on the upper branch of the curve $y^2 = 4x$ so that x is decreasing at the rate of 2 ins. per sec.   Find how fast the line through $A(2,0)$ and P is rotating at the point $(9,6)$.

36. If $x = 3t^2 - 6t + 1$, $y = 2t + 5$ are the equations of the path of a moving particle, where t is time, find when the velocity is least and show that the particle is at the vertex of the parabolic path at that time.

37. Find the volume of the solid generated by revolving the area bounded by $y = \sin x$, the x-axis and $4x = \pi$ about the line $y = 1$.

38. Differentiate $y = \sin x$, where x is expressed in degrees.

39. Differentiate $\sin x$ with respect to $x + \sqrt{x + 1}$.

40. If $y = a \sin x + b \cos x$, prove that $\dfrac{d^2y}{dx^2} + y = 0$.

41. If $y = \arcsin^2 x$, prove that $(1 - x^2) \dfrac{d^2y}{dx^2} - x \dfrac{dy}{dx} = 2$.

42. If $x = a(\theta - \sin \theta)$, $y = a(1 - \cos \theta)$, find $\dfrac{d^2y}{dx^2}$.

CHAPTER VIII

EXPONENTIAL AND LOGARITHMIC FUNCTIONS

69. Exponential Functions.

The quantity a^n defines a real number for all rational values of n, if a is a positive constant. In order that the laws governing the use of exponents as presented in algebra be available, they are stated as follows:

$$a^n = 1, \quad \text{if} \quad n = 0.$$
$$a^m \cdot a^n = a^{m+n}.$$
$$\frac{a^m}{a^n} = a^{m-n} = \frac{1}{a^{n-m}}.$$
$$(a^n)^p = a^{pn}.$$
$$a^{n/m} = \sqrt[m]{a^n}, \quad m \neq 0.$$

The equation

$$y = a^x,$$

where a is any constant, defines a function of x which is called an *exponential function*. If it is assumed that a is always positive and real, then the function is positive and real for all real values of x. Moreover, the function is one-valued and continuous for all those values of x.

Consider the exponential equation

$$y = 2^x.$$

As x becomes large without limit, y becomes large without limit. For $x = 0$, $y = 1$. For negative values of x, we may study the function

$$y = \frac{1}{2^x},$$

for the corresponding positive values of x. As x becomes large without limit in this function, y approaches zero. Hence, the x-axis is an asymptote of the curve. The curve is drawn in Figure 61.
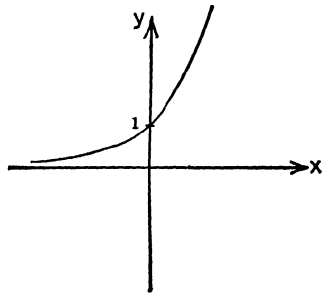
FIG. 61

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Considerable care should be exercised when exponential functions are plotted where limiting values are to be found. Such functions may not only possess discontinuities, but the behavior may be quite different from that of any function yet encountered.

Consider the exponential equation

$$y = 2^{1/x}.$$

The function is studied for positive values of x as follows: As x becomes large without limit, the exponent approaches zero and y approaches unity. As x approaches zero, the exponent becomes large without limit so that y increases without limit. For negative values of x, the function

$$y = \frac{1}{2^{1/x}}$$

is studied for positive values of x as follows: As x becomes large without limit, the exponent approaches zero and y approaches unity. As x approaches zero, the exponent and the denominator become infinite and, consequently, y approaches zero. In conclusion, the function is positive for all values of x, the curve is asymptotic to the line $y = 1$ and to the y-axis on the right and the curve terminates at the origin from the left. The curve is drawn in Figure 62.
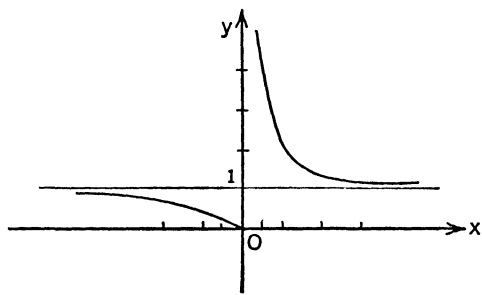
Fig. 62

Exercise 46

Draw each of the following curves.

1. $y = 3^x$.
2. $y = 2^{-x}$.
3. $y = (\frac{1}{3})^x$.
4. $y = 2^{1-x}$.
5. $y = 3^{x-1}$.

6. $y = 2^{-1/x}$.
7. $y = 2^{(x+1)/x}$.
8. $y = \dfrac{1}{2^{1+(1/x)}}$.
9. $y = x2^x$.
10. $y = 2^{\sin x}$.

70. Logarithmic Functions.

The logarithm of a number N is the exponent p of a constant b, called the base of the logarithm, required to give the number N. Thus,

$$\log_b N = p.$$

From the definition, this equation is equivalent to the one in exponential form,

$$b^p = N.$$

Logarithms, being exponents, obey the same laws of combination as do exponents. These laws may be written as follows:

$$\log_b N = 0, \quad \text{if } N = 1.$$

$$\log_b (MN) = \log_b M + \log_b N.$$

$$\log_b \left(\frac{M}{N}\right) = \log_b M - \log_b N.$$

$$\log_b (N)^k = k \log_b N,$$

where k is a real number.

Common and Natural Logarithms. While any positive number b, except 0 and 1, may be used as the base of a system of logarithms, in practice there are but two numbers used. For the first system, $b = 10$, which gives *common logarithms* of numbers. These are used in numerical computation, multiplication, division, extracting roots, etc. For the second system, the base is chosen as a number represented by e which arises in the differentiation given in Section 72. If the number e is used as a base, the system of logarithms is known as the *natural logarithms*.

In writing the logarithm of a number it is customary to omit the base for both common and natural logarithms. Thus,

$$\text{if } b = 10, \quad \log N = p \text{ is equivalent to } 10^p = N,$$

and $\quad$ if $b = e, \quad \ln N = p$ is equivalent to $\quad e^p = N$.

The logarithm of a number to any other base must express that base. For example,

$$\log_2 8 = 3, \quad \text{since } 2^3 = 8.$$

The equation

$$y = \log_b x$$

defines a function of x which is called a *logarithmic function*. This function is one-valued and continuous for all *positive* values of x, there being no logarithms of zero and negative numbers.

Consider the logarithmic equation

$$y = \log_3 x.$$

For values of x less than 1, y is negative and decreases without limit as x approaches zero. For $x = 1$, $y = 0$. For values of x greater than 1, y is positive and increases indefinitely as x increases. The curve is drawn in Figure 63.
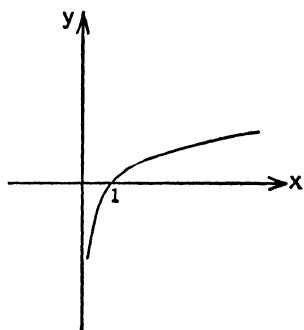
Change of Base. It is often desirable to change the base in a logarithmic equation from one constant to another. This may arise when a table of logarithms is being used which is expressed in terms of a base which is different from that used in an equation. Suppose that we have the equation

$$y = \log_b x$$

and that we wish to express the function in terms of a base a. From the definition of a logarithm,

$$b^y = x.$$

Fig. 63

Taking the logarithm of both sides of the equation to the desired base,

$$y \log_a b = \log_a x.$$

Hence,
$$y = \log_b x = \frac{\log_a x}{\log_a b}.$$

Exercise 47

GROUP A

Solve for x in each of the following equations.

1. $\log_4 x = \frac{1}{2}$.
2. $\log_9 x = -\frac{3}{2}$.
3. $\log_3 (\frac{1}{9}) = x$.
4. $\log_{27} 9 = x$.

5. $\log_x 27 = 3$.
6. $\log_x (\frac{1}{4}) = 4$.
7. $\log_b x = 3 \log_b 2 + 2 \log_b 3$.
8. $2 \log x = 3 \log 2 - \log 3$.

Draw each of the following curves.

9. $y = \log_2 x$.
10. $y = \log_2 3x$.

11. $y = \log (x - 1)$.
12. $y = 1 - \log x$.

GROUP B

13. Show that $\log_b b^x = x$.
14. Show that $b^{\log_b x} = x$.
15. Prove that $\log_b a = \dfrac{1}{\log_a b}$.

Solve for x in each of the following equations.

16. $y = 10^{2x}$.

17. $y = \tan 2^x$.

18. $y = \log \tan 2x$.

19. $y = \cos 2^{-x}$.

20. $y = 2^{x^2+1}$.

21. $y = \log \sin x$.

22. $y = \arctan 2^x$.

23. $y = 2^{x-\log_2 v}$.

24. $2 \log x = \log 2 + \log 8 - 4 \log 3 + 2$.

25. $2 \log_4 x = 3 - 2 \log_4 2 - 2 \log_4 5$.

Draw each of the following curves.

26. $y = x + \log x$.

27. $y = \log \sin x$.

28. $y = \log \cos x$.

29. $y = \log \tan x$.

30. $3^y = x$.

71. The Limit e.

In the derivation of the derivative of a logarithm we shall need to find the limit of an expression having the form

$$\lim_{z \to \infty} \left(1 + \frac{1}{z}\right)^z.$$

The rigorous proof that this limit exists is beyond the scope of our treatment. For our purposes, however, it is essential to show that the *limit exists* and to find a rough approximation to its value. For simplicity, we shall confine our treatment to the case in which z becomes infinite through positive integral values only.

If z is a positive integer, the expression can be expanded by the binomial theorem as follows:

$$\left(1 + \frac{1}{z}\right)^z = 1 + z\frac{1}{z} + \frac{z(z-1)}{2!}\frac{1}{z^2} + \frac{z(z-1)(z-2)}{3!}\frac{1}{z^3} + \cdots + \frac{1}{z^z}.$$

$$= 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{z}\right) + \frac{1}{3!}\left(1 - \frac{1}{z}\right)\left(1 - \frac{2}{z}\right)$$

$$+ \frac{1}{4!}\left(1 - \frac{1}{z}\right)\left(1 - \frac{2}{z}\right)\left(1 - \frac{3}{z}\right) + \cdots + \frac{1}{z^z}.$$

$$= 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \cdots$$

$$+ \left(\text{Terms in } \frac{1}{z}, \frac{1}{z^2}, \frac{1}{z^3}, \cdots, \frac{1}{z^z}\right).$$

From this

$$\lim_{z \to \infty} \left(1 + \frac{1}{z}\right)^z = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \cdots + \lim_{z \to \infty} \epsilon,$$

where ϵ represents all terms containing $1/z$, $1/z^2$, $1/z^3$, $\cdots$.

It can be shown that the limit of ϵ is zero as z becomes infinite, and that

$$e = \lim_{z \to \infty}\left(1 + \frac{1}{z}\right)^z = \lim_{n \to \infty}\left(1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{(n-1)!} + \cdots\right).$$

From this series an approximation to the value of the limit may be obtained by computing the sum of the series. By the use of a table of reciprocals, the sum of the first ten terms yields a value of e correct to four decimal places. The value of e to seven decimal places is

$$e = 2.7182818 \cdots.$$

As indicated, this number is represented by the letter e. This designation is as generally and consistently used, as is the Greek letter π for the number $3.14159 \cdots$. The importance of the number e in the theory of logarithms is discussed in Section 73.

72. Exponential and Logarithmic Differentiation Formulas.

The formulas for the differentiation of the exponential and logarithmic functions are as follows, where u is any function of x which can be differentiated:

(20) $$\frac{d}{dx} \log_b u = \frac{1}{u} \log_b e \frac{du}{dx}.$$

(21) $$\frac{d}{dx} \ln u = \frac{1}{u}\frac{du}{dx}.$$

(22) $$\frac{d}{dx} a^u = a^u \ln a \frac{du}{dx}.$$

(23) $$\frac{d}{dx} e^u = e^u \frac{du}{dx}.$$

73. Derivation of Exponential and Logarithmic Differentiation Formulas.

To derive the formula for the differentiation of a logarithmic function, let

$$y = \log_b u,$$

where u is any function of x which can be differentiated and b is any positive real number greater than 1. Let x be given the increment Δx and let the corresponding increments of y and u be Δy and Δu, respectively. Then

$$y + \Delta y = \log_b (u + \Delta u)$$

and
$$\Delta y = \log_b (u + \Delta u) - \log_b u = \log_b \left(1 + \frac{\Delta u}{u} \right).$$

Dividing both sides of the equation by Δx and supplying Δu in both numerator and denominator of the second member,

$$\frac{\Delta y}{\Delta x} = \frac{\Delta u}{\Delta x} \frac{1}{\Delta u} \log_b \left(1 + \frac{\Delta u}{u} \right).$$

Even yet, the limit of both sides of the equation cannot be taken. However, if u is also supplied in both numerator and denominator of the second member,

$$\frac{\Delta y}{\Delta x} = \frac{\Delta u}{\Delta x} \frac{1}{u} \frac{u}{\Delta u} \log_b \left(1 + \frac{\Delta u}{u} \right)$$

and if the factor $u/\Delta u$ is written as an exponent, the limit can be evaluated. Then

$$\frac{\Delta y}{\Delta x} = \frac{\Delta u}{\Delta x} \frac{1}{u} \log_b \left(1 + \frac{\Delta u}{u} \right)^{u/\Delta u}.$$

Taking the limit as Δx approaches zero, and using the fact that Δu approaches zero,

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} \cdot \frac{1}{u} \cdot \lim_{\Delta x \to 0} \log_b \left(1 + \frac{\Delta u}{u} \right)^{u/\Delta u}.$$

The logarithm is a continuous function, which permits us to write

$$\lim_{\Delta x \to 0} \log_b \left(1 + \frac{\Delta u}{u} \right)^{u/\Delta u} = \log_b \lim_{\Delta x \to 0} \left(1 + \frac{\Delta u}{u} \right)^{u/\Delta u} = \log_b e.$$

The latter equality is made possible from Section 71, where it was shown that

$$\lim_{z \to \infty} \left(1 + \frac{1}{z} \right)^z = \lim_{\Delta u \to 0} \left(1 + \frac{\Delta u}{u} \right)^{u/\Delta u} = e.$$

Hence, we have

(20)
$$\frac{dy}{dx} = \frac{1}{u} \log_b e \frac{du}{dx}.$$

In formula (20) it is observed that the differentiation of a logarithmic function to the base b contains the constant factor

$$\log_b e,$$

where e is the number obtained as the limit of a sum in Section 71.   This constant factor may be avoided by *building a system of logarithms having a base e*, since

$$\log_e e = \ln e = 1.$$

This is the *sole justification* for the use of the base e in the natural, or the *Naperian* system of logarithms.

For the most part, we are not concerned with numerical calculations in the calculus.   However, when we are, tables of natural logarithms are available.   In addition, by the use of the relation

$$\ln N = \frac{\log N}{\log e} = \frac{\log N}{0.43429} = 2.30259 \log N,$$

natural logarithms of numbers may be readily converted to the common logarithms, and vice versa.

To find the formula for the differentiation of the function

$$y = \ln u,$$

the base b is replaced by e in formula (20).   Since $\ln e = 1$,

(21)
$$\frac{dy}{dx} = \frac{1}{u}\frac{du}{dx}.$$

To find the derivative of the exponential function, let

$$y = a^u,$$

where a is any positive real number and u is a function of x which can be differentiated.   Taking the natural logarithm of both sides of the equation,

$$\ln y = u \ln a.$$

Differentiating with respect to x,

$$\frac{1}{y}\frac{dy}{dx} = \ln a \frac{du}{dx}.$$

(22)
$$\frac{dy}{dx} = a^u \ln a \frac{du}{dx}.$$

To find the derivative of the particular exponential function

$$y = e^u,$$

let $a = e$ in formula (22).   Since $\ln e = 1$,

(23)
$$\frac{dy}{dx} = e^u \frac{du}{dx}.$$

The applications of the formulas derived are illustrated by the differentiation of the following functions:

If
$$y = \log(x^2 - 4),$$
$$\frac{dy}{dx} = \frac{2x}{x^2 - 4} \log e.$$

If
$$y = \ln \sin ax,$$
$$\frac{dy}{dx} = a \frac{\cos ax}{\sin ax} = a \cot ax.$$

If
$$f(x) = 6^{\cos 2x},$$
$$f'(x) = -2 \ln 6 \sin 2x \, 6^{\cos 2x}.$$

If
$$f(x) = e^{\arctan bx},$$
$$f'(x) = \frac{b}{1 + b^2 x^2} e^{\arctan bx}.$$

Exercise 48

GROUP A

Differentiate each of the following functions

1. $y = \log_2 (x^2 + 1)$.

2. $y = \ln (x^2 + 4x + 2)$.

3. $y = 2^{x^2}$.

4. $y = e^{-1/x}$.

5. $y = \frac{1}{4} \ln \frac{x - 2}{x + 2}$.

6. $y = e^x \sin 3x$.

7. $y = a(e^{x/a} + e^{-x/a})$.

8. $y = \log (2 - x)$.

9. $y = \ln(e^{2x} + e^{-2x})$.

10. $y = x \arctan x - \ln \sqrt{1 + x^2}$.

11. $y = \ln \frac{1 - \sin x}{1 + \sin x}$.

12. $y = \ln x(x^2 - 1)$.

13. $y = \frac{\ln x}{x}$.

14. $y = e^{\sin^2 x}$.

15. $y = \log \tan^2 x$.

16. $y = 3^{\cos^2 x}$.

17. Find the equation of the tangent to the curve $y = e^x$ at $x = 1$.

18. Find the equation of the tangent to the curve $y = \ln x$ at $x = e$.

GROUP B

Find the equations of the tangents to each of the following curves.

19. $y = \ln x$ at $x = 1$.

20. $y = e^x$ having slope 3.

21. $y = \ln x$ parallel to the line $x - 4y + 4 = 0$.

22. $y = x \ln x$ perpendicular to the line $x + 3y = 0$.

23. $y = 2^x$ at $x = 3$.

24. $y = \log x$ at $x = 10$.

25. Differentiate $y = x^x$ with respect to x by first taking the logarithm of both sides of the equation.

26. Derive the formula for the differentiation of $y = u^v$ with respect to x, where u and v are functions of x which can be differentiated.

Find $\dfrac{dy}{dx}$ for each of the following equations.

27. $y = \ln^2 2x$.

28. $y = 3^{\arctan^2 x}$.

29. $y = e^{\sin e^{2x}}$.

30. $y = \log^2 ax$.

31. $y = \ln \ln x$.

32. $y = \ln \cos^2 e^x$.

33 $x = e^{-\theta}, \quad y = e^{2\theta+1}$.

34. $x = \log z \quad y = e^z$.

35. $y = \ln^2 (xe^x)$.

36. $y = \ln \ln \sin ax$.

37. $y = \dfrac{1}{\ln ax}$.

38. $y = \ln \dfrac{\sqrt{x+1}-1}{\sqrt{x+1}+1}$.

39. $e^{x+y} = \ln x/y$.

40. $xy = \ln (x+y)$.

41. $e^{x^2-y^2} = axy$.

42. $x^2 + y^2 = 2\log_b (x+y)$.

74. Tracing Curves of Exponential Functions.

The graphs of the simpler exponential functions which are drawn in Figures 61 and 62 have no maximum and minimum points nor inflection points. If this section we wish to consider those functions whose graphs possess such points.

Consider the function

$$f(x) = x^3 e^{-x}.$$

The first derivative and the second derivative are

FIG. 64

$$f'(x) = x^2(3-x)e^{-x},$$

and

$$f''(x) = x(x^2 - 6x + 6)e^{-x}.$$

If $f'(x) = 0$, $x = 0$ and $x = 3$.

If $f''(x) = 0$, $x = 0$,

$x = 3 - \sqrt{3}$ and $x = 3 + \sqrt{3}$.

The second derivative test shows that $B(3, 27e^{-3})$ is a maximum point. The points O, A and C having abscissas 0, $3 - \sq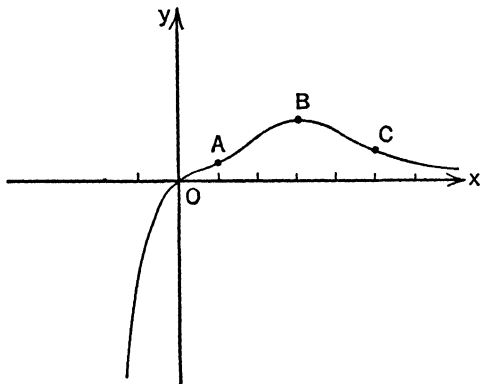rt{3}$ and $3 + \sqrt{3}$, respectively, are inflection points. The curve is concave downward in the intervals to the left of the origin and between A and C. It is concave upward in the intervals between O and A and to the right of C. The curve is drawn in Figure 64.

Exercise 49

GROUP A

Find the coordinates of any maximum, minimum and inflection points and draw each of the following curves. Give the intervals over which each curve is concave upward and downward.

1. $y = xe^x$.

2. $y = xe^{-2x}$.

3. $y = x \sin x$.

4. $y = \dfrac{e^x}{x}$.

5. Show that the function $y = \ln x$ is an increasing function for all positive values of x and that the curve is everywhere concave downward

6. Show that the function $y = e^x$ is an increasing function for all values of x and that the curve is concave upward everywhere.

7. Find d^4y/dx^4 for $y = e^{ax}$.

8. Find d^6y/dx^6 for $y = \ln x$.

GROUP B

Draw each of the following curves.

9. $y = xe^{-2x^2}$.

10. $y = xe^{(x+1)/x}$.

11. $y = x^2e^{-x}$.

12. $y = e^{2x} + e^{-2x}$.

13. Find d^5y/dx^5 for $y = 2^x$.

14. Find d^7y/dx^7 for $y = \log_b ax$.

15. Find approximately the value of $\ln 2\ 01$ from $\ln 2$.

16. Find approximately the value of $e^{2\ 01}$ from e^2.

17. Find the equation of the tangent to the curve $y = \ln \sin x$ at $x = \pi/6$.

18. Show that the curve $y = \ln \sin 2x$ is everywhere concave downward.

19. Draw the curve $y = \ln \cos 2x$.

20. Write the equation of the tangent to the curve $y = \ln \tan x$ at the inflection point $0 < x < \pi/2$.

75. Integration.

By reversing the formulas for the differentiation of exponential and logarithmic functions, the formulas for the integration of those functions can be obtained. Thus

(13) $$\int \frac{du}{u} = \ln u + C.$$

(14) $$\int a^u \, du = \frac{1}{\ln a} a^u + C.$$

(15) $$\int e^u \, du = e^u + C.$$

In each case, u is a function of a variable and du is the differential of that function.

If in formula (13) $u = x$, $du = dx$ and

$$\int \frac{dx}{x} = \ln x + C.$$

This is the exceptional case, where $n = -1$, of formula (1) as presented in Section 29 for the integration

$$\int x^n\, dx = \frac{1}{n+1} x^{n+1} + C.$$

The following integrations are carried out in illustration of the applications of the formulas:

$$\int \frac{dx}{x-4} = \ln(x-4) + C.$$

$$\int \frac{x\, dx}{x^2+1} = \frac{1}{2} \int \frac{2x\, dx}{x^2+1} = \frac{1}{2}\ln(x^2+1) + C$$
$$= \ln\sqrt{x^2+1} + C.$$

$$\int_1^2 \frac{2x-2}{x^2-2x+3}\, dx = \ln(x^2-2x+3)\Big]_1^2 = \ln\frac{3}{2},$$

$$\int 2^{2x}\, dx = \frac{1}{2}\int 2^{2x}(2\, dx) = \frac{1}{\ln 4} 2^{2x} + C.$$

$$\int_0^2 xe^{x^2}\, dx = \frac{1}{2}\int_0^2 e^{x^2}(2x\, dx) = \frac{1}{2}e^{x^2}\Big]_0^2 = \frac{e^4-1}{2}.$$

To find the area bounded by the coordinate axes, the line $x - 4 = 0$ and the curve

$$y = 2^x,$$

we proceed as follows: The element of area is

$$dS = y\, \Delta x = 2^x\, \Delta x.$$

Summing the elements, taking the limit of the sum and applying the fundamental theorem,

$$S = \int_0^4 2^x\, dx = \frac{2^x}{\ln 2}\Big]_0^4$$

$$S = \frac{16-1}{\ln 2} = \frac{15}{\ln 2}.$$

To find the volume generated by the rotation of the area bounded by the coordinate axes, the line $x - 3 = 0$ and the curve

$$y = e^x$$

about the x-axis, we proceed as follows: The element of volume is

$$dV = \pi y^2 \, \Delta x = \pi e^{2x} \, \Delta x.$$

As before,

$$V = \pi \int_0^3 e^{2x} \, dx = \frac{\pi}{2}(e^6 - 1).$$

Exercise 50

GROUP A

Carry out each of the following indicated integrations.

1. $\int \frac{dx}{2x}$. $\Rightarrow \frac{1}{2} \log x$ 5. $\int e^{x-1} \, dx$. $\frac{e}{e}$

2. $\int e^{2x} \, dx$. $\frac{e^{2x}}{2}$ 6. $\int 2^{x+1} \, dx$.

3. $\int 2^{x+1} \, dx$. $2 \cdot 2^x = 2 \cdot \frac{2^x}{\log 2}$ 7. $\int \frac{x \, dx}{x^2 - 1}$.

4. $\int \frac{dx}{x + 2}$. $= \log(x+2)$ 8. $\int x e^{2x^2} \, dx$.

9. The velocity of a particle moving on a line at any time is given by $v = 2/t$. If $s = 2$ when $t = 1$, find s when $t = 4$.
10. The slope of a curve at any point is $2/x$. If the curve passes through the point $(2,3)$, find its equation.
11. Find the area under the curve $y = e^x$ from $x = 0$ to $x = 4$.
12. Find the area under the curve $xy = 4$ from $x = 1$ to $x = 3$.

GROUP B

Carry out each of the following indicated integrations.

13. $\int x 2^{x^2} \, dx$.   16. $\int \frac{e^x}{e^x + 1} \, dx$.

14. $\int (x + 1)e^{x^2+2x} \, dx$.   17. $\int e^x \sin e^x \, dx$.

15. $\int \frac{x + 3}{x^2 + 6x + 1} \, dx$.   18. $\int \frac{\cos x}{1 + \sin x} \, dx$.

19. Find the area under the curve $y = e^{x+1}$ from $x = -2$ to $x = 2$.
20. Find the area under the curve $y = 4^x$ from $x = -4$ to $x = 1$.
21. Find the area under the curve $y = 2(e^{x/2} + e^{-x/2})$ from $x = -2$ to $x = 2$.
22. Find the area between $xy = 8$ and $x + y = 6$.
23. The slope of a curve at any point is equal to three times the slope of the line from the point to the origin. If the curve passes throught the point $(2,4)$, find its equation.
24. Find the volume generated by the rotation about the x-axis of the area bounded by $xy = 1$, the x-axis, $x = 1$ and $x = 4$.

25. Find the volume generated by the rotation about the x-axis of the area bounded by $y = e^{2x}$, the x-axis, the y-axis and $x = 3$.

26. Find the volume generated by the rotation about the x-axis of the area bounded by $y = 2(e^{x/2} + e^{-x/2})$, the x-axis, $x = -2$ and $x = 2$.

27. Find the volume generated by the rotation about the line $y + 1 = 0$ of the area bounded by $xy = 4$, the x-axis, $x = 2$ and $x = 4$.

Differentiate each of the following functions.

28. $\ln \dfrac{e^{2x} + 1}{e^{2x} - 1}$

29. $\log_b \dfrac{\sqrt{x + 4} + 2}{\sqrt{x + 4} - 2}$

30. $a^{\arctan^2 ax}$

31. e^{e^x}

32. $a^{bx^2 + cx + d}$

33. $b^{\sqrt{4 - x^2}}$

76. Law of Natural Growth.

An important application of exponential functions occurs in problems where the rate of change of a quantity with respect to a variable is proportional to the quantity itself. If y represent the quantity and the derivative with respect to x, its rate of change, the mathematical statement of the law is

$$\frac{dy}{dx} = ky,$$

where k is the proportionality factor. This type of rate of change gives rise to the equation known as the *law of natural growth*. This equation is obtained as follows:

$$dy = ky \, dx, \quad \text{or} \quad \frac{dy}{y} = k \, dx.$$

Integrating,

$$\ln y = kx + C.$$

From the definition of a natural logarithm,

$$y = e^{kx+C} = e^C \, e^{kx}.$$

Replacing the constant e^C by A for convenience,

$$y = Ae^{kx}.$$

This law obtained its name from the fact that, for example, the natural growth of bacteria in a culture follows the type of change this equation expresses. Suppose that the population of a culture of 1000 increases to

50,000 in 10 hours. We find its population at the end of 20 hours as follows:

$$\frac{dN}{dt} = kN, \quad \ln N = kt + C$$

$$N = Ae^{kt}.$$

If $t = 0$,

$$N = A = 1000 \quad \text{and} \quad N = 1000\, e^{kt}.$$

If $t = 10$,

$$50,000 = 1000\, e^{10k}, \quad e^k = (50)^{1/10}.$$

Hence, if $t = 20$,

$$N = 1000\, (50)^2 = 2,500,000.$$

Other variations in which the law of natural growth apply are that the rate of change of air pressure with respect to the distance from the surface of the earth is proportional to the pressure at each distance from the surface, that the rate of change of the difference of the temperature of a body and a cooling flow of a medium is proportional to that difference and that the rate of change of an amount of money put at interest and compounded continuously is proportional to the principal.

In illustration of the latter variation, suppose that $100 is placed at interest compounded continuously at the rate of 4 per cent. Let us find the number of years required to make the principal $500. Since

$$\frac{dP}{dt} = 0.04\, P,$$

$$\ln P = 0.04t + C, \quad P = Ae^{0.04t}.$$

If $t = 0$, $\qquad P = A = 100, \qquad P = 100\, e^{0.04t}.$

If $P = 500$, $\qquad e^{0.04t} = 5 \qquad 0.04t = \ln 5.$

Hence, $\qquad t = 40.2$ yrs.,

using a table of natural logarithms.

Exercise 51

GROUP A

1. The rate of change of y with respect to x is always equal to 10 times y. If $y = 20$ when $x = 0$, find the equation connecting x and y.
2. The rate of change of y with respect to x is proportional to y. If $y = 6$ when $x = 0$ and $y = 12$ when $x = 3$, find the equation connecting x and y.

3. The rate of change of y with respect to t is always proportional to y. If $y = 4$ when $t = 2$ and $y = 9$ when $t = 4$, find the equation connecting y and t.

4. The sum of \$200 is put at interest at the rate of 5 per cent per year. If the interest is compounded continuously, find the amount in 20 years.

5. The population of a locality is 20,000 and is increasing at a rate proportional to the population at the time. Find the population fifty years later, if the population is 50,000 twenty-five years later.

6. There are 1000 bacteria in a culture and 5 hrs. later there are 5000. Find the number of hours after which there will be 10,000 bacteria.

7. A soluble substance in solution is being decomposed at a rate proportional to the amount present. If 50 lbs. is reduced to 20 lbs. in 15 hrs., after how long a time will there be 10 lbs. remaining?

8. In a chemical reaction the rate of change of concentration is proportional to the concentration at any time. If the concentration is 0.02 when $t = 0$ and is 0.01 when $t = 5$, find the concentration when $t = 10$.

GROUP B

9. Resistance is applied to a rotating wheel so that the angular acceleration is proportional to the angular velocity. If the wheel is making 100 revolutions per sec. in the beginning, and in 1 min. it is reduced to 50 revolutions per sec., find the time necessary to reduce it to 10 revolutions per sec.

10. The population of a city changes at a rate proportional to itself. If it is now 50,000 and 25 years ago it was 20,000, find the population 15 years hence.

11. A particle moves on a curve in such a way that the rate of change of the ordinate with respect to the abscissa is proportional to the ordinate. Find the equation of the curve, if the slope of the curve is $-3/2$ at the point $(2,3)$.

12. The rate of change of the slope of a curve at every point is equal to twice the slope of the curve. If the slope is 3 at $(0,3)$, find the equation of the curve.

13. Assume the compound interest law $A = P\left(1 + \dfrac{r}{n}\right)^{nt}$, where A is the amount, P the principal, r the annual interest rate, t the number of years and n the number of compoundings per year. Derive the natural growth law from it, if the interest is compounded continuously.

14. An electric current decreases in intensity 20 per cent in 20 minutes. After how long will it have one-hundredth part of its original intensity?

Verify each of the following integrations.

15. If $x\dfrac{dy}{dx} = y + 2$, $\quad Ax - y - 2 = 0$.

16. If $xy\dfrac{dy}{dx} = y^2 + 3$, $\quad Ax^2 - y^2 = 3$.

17. If $xy\dfrac{dy}{dx} = \sqrt{y^2 + 2}$, $\quad x = Ae^{\sqrt{y^2+2}}$.

GROUP C

18. Differentiate $y = x^{\sin x}$.

19. Show that the curve $y = 3x + \sin x - \cos x$ has no maximum or minimum points and find the coordinates of one inflection point.

20. Draw the curve $y = \cos^3 2x$, from $x = -\pi$ to $x = \pi$.

21. Find the approximate difference between $y = \ln x$ and $y = \ln (x + \Delta x)$ for large values of x.

22. Find the approximate difference between a^x and $a^{x+\Delta x}$ for large negative values of x.

23. The inclination of a pendulum to the vertical is given by $\theta = ae^{-kt} \cos (nt + b)$, when the resistance of the air is taken into account. Show that the greatest inclinations occur at equal intervals of time π/n.

24. Find the minimum value of $y = ae^{nx} + be^{-nx}$.

25. Find the coordinates of the minimum point and the inflection point and draw the curve $y \ln x = x$.

26. Show that the maximum rectangle with one side on the x-axis which can be inscribed under the curve $y = e^{-x^2}$ has two of its vertices at the points of inflection.

CHAPTER IX

INTEGRATION

77. Standard Formulas for Integration.

This chapter is devoted, primarily, to the technique of integration. As a first step in acquiring facility in integrating various expressions, called *integrands*, familiarity with the formulas which have been used in previous chapters is essential. For purposes of reference these formulas are collected in this section, some of which are changed slightly. These changes are discussed in the sections which follow. In addition, other formulas are given which are necessary for the purposes of this text.

$$(1) \qquad \int x^n \, dx = \frac{1}{n+1} x^{n+1} + C, \quad \text{where } n \neq -1.$$

$$(2) \qquad \int au \, du = a \int u \, du, \quad \text{where } a \text{ is a constant.}$$

$$(3) \qquad \int (u + v) \, dx = \int u \, dx + \int v \, dx, \quad \begin{cases} \text{where } u \text{ and } v \text{ are} \\ \text{functions of } x. \end{cases}$$

$$(4) \qquad \int \sin u \, du = -\cos u + C.$$

$$(5) \qquad \int \cos u \, du = \sin u + C.$$

$$(6) \qquad \int \sec^2 u \, du = \tan u + C.$$

$$(7) \qquad \int \csc^2 u \, du = -\cot u + C.$$

$$(8) \qquad \int \tan u \sec u \, du = \sec u + C.$$

$$(9) \qquad \int \cot u \csc u \, du = -\csc u + C.$$

$$(10) \qquad \int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin \frac{u}{a} + C.$$

$$(11) \qquad \int \frac{du}{a^2 + u^2} = \frac{1}{a} \arctan \frac{u}{a} + C.$$

(12) $\quad \displaystyle\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a} \operatorname{arcsec} \frac{u}{a} + C.$

(13) $\quad \displaystyle\int \frac{du}{u} = \ln u + C.$

(14) $\quad \displaystyle\int a^u du = \frac{1}{\ln a} a^u + C.$

(15) $\quad \displaystyle\int e^u \, du = e^u + C.$

(16) $\quad \displaystyle\int u^n \, du = \frac{1}{n+1} u^{n+1} + C, \quad \text{where } n \neq -1.$

(17) $\quad \displaystyle\int \tan u \, du = \ln \sec u + C.$

(18) $\quad \displaystyle\int \cot u \, du = \ln \sin u + C.$

(19) $\quad \displaystyle\int \sec u \, du = \ln (\sec u + \tan u) + C.$

(20) $\quad \displaystyle\int \csc u \, du = \ln (\csc u - \cot u) + C.$

(21) $\quad \displaystyle\int u \, dv = uv - \int v \, du.$

78. Integral of $u^n \, du$

The integral of a function to any exponent except -1, is given by formula

(16) $$\int u^n \, du = \frac{1}{n+1} u^{n+1} + C.$$

The single case in which $n = -1$ is an application of formula

(13) $$\int \frac{du}{u} = \ln u + C.$$

In the application of these formulas, it is most important to know that *u represents any function of a variable and that du represents its differential.* In the integrations carried out in previous sections u represented, for the most part, a single variable. In this section the functions which u represent are varied; they may be algebraic, trigonometric or exponential.

The first task in an integration is to separate the integrand into two parts, that which is represented by u and that which is represented by du. Having chosen the function u, the remaining part of the integrand must differ from its differential du at most by a constant. As a corollary of formula

$$(2) \qquad \int au \, du = a \int u \, du,$$

a constant may be introduced in the integrand, provided that its reciprocal is placed before the integration sign.

The following illustrate the applications of formulas (13) and (16):

$$\int \frac{(x-1) \, dx}{\sqrt{x^2 - 2x + 3}} = \frac{1}{2} \int \frac{(2x-2) \, dx}{\sqrt{x^2 - 2x + 3}} = \sqrt{x^2 - 2x + 3} + C,$$

where $u = x^2 - 2x + 3$, $du = (2x - 2) \, dx$ and $n = -\frac{1}{2}$.

$$\int \frac{(x-1) \, dx}{x^2 - 2x + 3} = \frac{1}{2} \int \frac{(2x-2) \, dx}{x^2 - 2x + 3} = \ln \sqrt{x^2 - 2x + 3} + C,$$

where $n = -1$.

$$\int \frac{x^2 \, dx}{\sqrt{(a^2 - x^3)^3}} = -\frac{1}{3} \int (a^2 - x^3)^{-3/2}(-3x^2 \, dx) = \frac{2}{3\sqrt{a^2 - x^3}} + C,$$

where $u = a^2 - x^3$, $du = -3x^2 \, dx$ and $n = -\frac{3}{2}$.

$$\int \frac{\sin 2x \, dx}{\sqrt{1 + \cos 2x}} = -\frac{1}{2} \int (1 + \cos 2x)^{-1/2}(-\sin 2x)(2 \, dx)$$
$$= -\sqrt{1 + \cos 2x} + C,$$

where $u = 1 + \cos 2x$, $du = -2 \sin 2x \, dx$ and $n = -\frac{1}{2}$.

$$\int \frac{\sin^2 3x}{\sec 3x} \, dx = \frac{1}{3} \int \sin^2 3x \cos 3x \, 3 \, dx = \frac{1}{9} \sin^3 3x + C,$$

where $u = \sin 3x$, $du = 3 \cos 3x \, dx$ and $n = 2$.

$$\int \frac{1 + \cos 3x}{3x + \sin 3x} \, dx = \frac{1}{3} \int \frac{3 + 3 \cos 3x}{3x + \sin 3x} \, dx = \frac{1}{3} \ln (3x + \sin 3x) + C,$$

where $u = 3x + \sin 3x$, $du = 3(1 + \cos 3x) \, dx$ and $n = -1$.

$$\int \frac{\ln^3 (3x + 1)}{3x + 1} \, dx = \frac{1}{3} \int \ln^3 (3x + 1) \frac{3 \, dx}{3x + 1} = \frac{1}{12} \ln^4 (3x + 1) + C,$$

where $u = \ln (3x + 1)$, $du = \dfrac{3\,dx}{3x + 1}$ and $n = 3$.

$$\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx = \ln (e^x + e^{-x}) + C,$$

where $u = e^x + e^{-x}$, $du = (e^x - e^{-x})\,dx$ and $n = -1$.

$$\int \frac{a^x\,dx}{(1 - a^x)^2} = -\frac{1}{\ln a} \int (1 - a^x)^{-2}(- a^x \ln a)\,dx = \frac{1}{\ln a} \cdot \frac{1}{(1 - a^x)} + C$$

where $u = 1 - a^x$, $du = - a^x \ln a\,dx$ and $n = -2$.

If the integrand assumes the form of a rational fraction whose *numerator is of the same or higher degree than that of the denominator, the indicated division should be performed* and the result integrated term by term. For example,

$$\int \frac{x^3 + x^2 - 4x + 3}{x - 1}\,dx = \int \left(x^2 + 2x - 2 + \frac{1}{x - 1} \right) dx$$

$$= \frac{x^3}{3} + x^2 - 2x + \ln (x - 1) + C.$$

The integrals which are given in the exercises following any particular section of this chapter, are not necessarily to be evaluated by the formulas and methods considered in that particular section. As more and varied integrands are set for evaluation and as new methods are presented, the wider becomes the choice of a method of attack. While many integrals can be evaluated in but one way, others can be done by various methods. Thus, the more miscellaneous is the group of integrands, the more it becomes an exercise in the recognition of types.

Exercise 52

GROUP A

Evaluate each of the following integrals.

1. $\displaystyle\int \frac{(x^2 + 1)^2}{x^3}\,dx$.

2. $\displaystyle\int \frac{x^3}{x - 1}\,dx$.

3. $\displaystyle\int x^2(x^3 + 2)^4\,dx$.

4. $\displaystyle\int x\sqrt{x^2 + 2}\,dx$.

5. $\displaystyle\int \frac{x^2}{\sqrt{x^3 + 2}}\,dx$.

6. $\displaystyle\int \frac{3\,dx}{(1 + 2x)^3}$.

7. $\displaystyle\int \frac{x + 1}{\sqrt{x^2 + 2x}}\,dx$.

8. $\displaystyle\int \frac{x + 1}{x^2 + 2x + 4}\,dx$.

9. $\int \dfrac{x^2}{4x^3 - 3} \, dx.$

10. $\int 8x^2 \sqrt[3]{x^3 + 4} \, dx.$

11. $\int \dfrac{4x}{\sqrt[3]{4 - x^2}} \, dx.$

12. $\int \dfrac{6x}{x^2 + 1} \, dx.$

13. $\int \dfrac{2 \, dx}{\sqrt{1 - x^2}}.$

14. $\int \sqrt{x}(1 - x^2) \, dx.$

15. $\int \dfrac{x^2 \, dx}{(3 + 4x^3)^2}.$

16. $\int \sin x \cos x \, dx.$

17. $\int \cos^2 2x \sin 2x \, dx.$

18. $\int \dfrac{\cos x}{\sin^2 x} \, dx.$

19. $\int \dfrac{\sin ax}{\cos^3 ax} \, dx.$

20. $\int \tan^2 x \sec^2 x \, dx.$

21. $\int \sin^4 3x \cos 3x \, dx.$

22. $\int \sec^3 2x \tan 2x \, dx.$

23. $\int \cot bx \csc^2 bx \, dx.$

24. $\int \dfrac{1 + \cos 2x}{(2x + \sin 2x)^2} \, dx.$

25. $\int \dfrac{\sec^2 ax}{1 + \tan ax} \, dx.$

26. $\int \dfrac{\sec^2 6x}{(1 + \tan 6x)^2} \, dx.$

27. $\int e^x(1 - e^x)^2 \, dx.$

28. $\int \dfrac{e^x}{1 - e^x} \, dx;$

29. $\int \dfrac{e^x}{(1 - e^x)^2} \, dx.$

30. $\int \dfrac{e^x}{\sqrt{1 - e^x}} \, dx.$

GROUP B

Evaluate each of the following integrals.

31. $\int \dfrac{x + 3}{x - 1} \, dx$

32. $\int \dfrac{(x - 1)^2}{3x} \, dx.$

33. $\int \dfrac{x^2 - 4}{x - 3} \, dx.$

34. $\int \dfrac{3x}{(2x^2 - 1)^2} \, dx.$

35. $\int \dfrac{x^3 + x}{x^2 + 2} \, dx.$

36. $\int (x^3 + 2)(x^2 + 1) \, dx.$

37. $\int \dfrac{3 \, dx}{1 + 4x^2}.$

38. $\int \dfrac{x}{4 - x^2} \, dx.$

39. $\int (2x^{3/2} + 1)^{2/3} \sqrt{x} \, dx.$

40. $\int \dfrac{dx}{x^2 + 4x + 4}.$

41. $\int \dfrac{x \, dx}{4x^4 + 4x^2 + 1}.$

42. $\int \dfrac{x}{\sqrt{1 - 8x^2}} \, dx.$

43. $\int \dfrac{6}{1 + 7x^2} \, dx.$

44. $\int \sqrt{\dfrac{1 - \sqrt{x}}{x}} \, dx.$

45. $\int \sec^4 ax \, dx.$

46. $\int \dfrac{\sin (3 - x)}{\sec^3 (3 - x)} \, dx.$

47. $\int \dfrac{e^x}{\sqrt{1 - e^{2x}}} \, dx.$

48. $\int \dfrac{e^x}{1 + e^{2x}} \, dx.$

49. $\displaystyle\int \sin^3 x \, dx.$

50. $\displaystyle\int x^2 \sin x^3 \, dx.$

51. $\displaystyle\int \frac{e^x + e^{-x}}{e^x - e^{-x}} \, dx.$

52. $\displaystyle\int \frac{e^x + \sin x}{\sqrt{e^x - \cos x}} \, dx.$

53. $\displaystyle\int \frac{e^{ax} - \cos ax}{e^{ax} - \sin ax} \, dx.$

54. $\displaystyle\int \frac{e^{ax} - e^{-ax}}{e^{ax} + e^{-ax}} \, dx.$

55. $\displaystyle\int x \sin^2 x^2 \cos x^2 \, dx.$

56. $\displaystyle\int \frac{\ln^2 x}{x} \, dx.$

57. $\displaystyle\int \frac{\sqrt{1 + \ln x}}{x} \, dx.$

58. $\displaystyle\int x \, 2^{x^2} \, dx.$

59. $\displaystyle\int \frac{e^{2x} + 1}{e^{2x} - 1} \, dx.$

60. $\displaystyle\int \sin^5 2x \, dx.$

79. Inverse Trigonometric Functions.

Formulas (10) and (11) for the integration of functions giving rise to the arcsine and the arctangent of a function, as given in Section 77, have slightly different forms than those obtained by the reversal of the corresponding differentiation formulas. To obtain these more general forms we proceed as follows:

(10)
$$\int \frac{du}{\sqrt{a^2 - u^2}} = \int \frac{\dfrac{du}{a}}{\sqrt{1 - \dfrac{u^2}{a^2}}} = \arcsin \frac{u}{a} + C.$$

(11)
$$\int \frac{du}{a^2 + u^2} = \frac{1}{a} \int \frac{\dfrac{du}{a}}{1 + \dfrac{u^2}{a^2}} = \frac{1}{a} \arctan \frac{u}{a} + C.$$

The two following integrations illustrate the use of these formulas.

$$\int \frac{dx}{\sqrt{7 + 12x - 4x^2}} = \frac{1}{2} \int \frac{2 \, dx}{\sqrt{16 - (2x - 3)^2}} = \frac{1}{2} \arcsin \frac{2x - 3}{4} + C.$$

$$\int \frac{dx}{9x^2 + 6x + 4} = \frac{1}{3} \int \frac{3 \, dx}{3 + (3x + 1)^2} = \frac{1}{3\sqrt{3}} \arctan \frac{3x + 1}{\sqrt{3}} + C.$$

The two following integrations illustrate the combinations of formulas (11) and (13) and of formulas (10) and (16).

$$\int \frac{x + 3}{x^2 - 4x + 8} \, dx = \frac{1}{2} \int \frac{2x - 4}{x^2 - 4x + 8} \, dx + 5 \int \frac{dx}{4 + (x - 2)^2}$$

$$= \frac{1}{2} \ln (x^2 - 4x + 8) + \frac{5}{2} \arctan \frac{x - 2}{2} + C.$$

$$\int \frac{2x + 3}{\sqrt{5 - 4x^2}}\, dx = -\frac{1}{4}\int \frac{-8x\, dx}{\sqrt{5 - 4x^2}} + \frac{3}{2}\int \frac{2\, dx}{\sqrt{5 - 4x^2}}$$

$$= -\frac{1}{2}\sqrt{5 - 4x^2} + \frac{3}{2}\arcsin \frac{2x}{\sqrt{5}} + C.$$

Exercise 53

GROUP A

Evaluate each of the following integrals.

1. $\displaystyle\int \frac{dx}{\sqrt{16 - 9x^2}}$.

2. $\displaystyle\int \frac{dx}{x^2 + 5}$.

3. $\displaystyle\int \frac{dx}{\sqrt{3 - 2x^2}}$.

4. $\displaystyle\int \frac{dx}{5x^2 + 3}$.

5. $\displaystyle\int \frac{dx}{x^2 + 2x + 2}$.

6. $\displaystyle\int \frac{dx}{\sqrt{4 - (x - 1)^2}}$.

7. $\displaystyle\int \frac{dx}{\sqrt{2x - x^2}}$.

8. $\displaystyle\int \frac{x + 1}{x^2 + 4}\, dx$.

9. $\displaystyle\int \frac{x + 1}{\sqrt{4 - x^2}}\, dx$.

10. $\displaystyle\int \frac{x + 3}{x^2 + 4x + 8}\, dx$.

11. $\displaystyle\int e^{4ax}\, dx$.

12. $\displaystyle\int xe^{ax^2}\, dx$.

13. $\displaystyle\int \frac{\sin x}{1 + \cos x}\, dx$.

14. $\displaystyle\int \frac{x + 2}{(x^2 + 4x + 2)^2}\, dx$.

15. $\displaystyle\int \frac{x + 2}{x^2 + 4x + 2}\, dx$.

16. $\displaystyle\int (1 + 3\sin^3 x)^2 \cos x\, dx$.

GROUP B

Evaluate each of the following integrals.

17. $\displaystyle\int \frac{x + 8}{x^2 - 4x + 7}\, dx$

18. $\displaystyle\int \frac{7}{\sqrt{4x - x^2 - 2}}\, dx$

19. $\displaystyle\int \frac{x - 1}{\sqrt{3 + 2x - x^2}}\, dx$.

20. $\displaystyle\int \frac{x + 2}{3x^2 + 2}\, dx$.

21. $\displaystyle\int \frac{x - 2}{\sqrt{2 - 3x^2}}\, dx$.

22. $\displaystyle\int \frac{x - 3}{x^2 + x + 1}\, dx$.

23. $\displaystyle\int \frac{x^2 + x + 4}{x^2 + x + 1}\, dx$.

24. $\displaystyle\int \frac{e^x}{e^{2x} + 2e^x + 5}\, dx$.

25. $\displaystyle\int \frac{e^{x/2}}{\sqrt{4 - 3e^x}}\, dx$.

26. $\displaystyle\int \frac{\sin x}{\cos x}\, dx$.

27. $\displaystyle\int \frac{\cos x}{\sin x}\, dx$. ✓

28. $\displaystyle\int e^{\sin ax} \cos ax\, dx$.

29. $\displaystyle\int a^{\cos x} \sin x\, dx$.

30. $\displaystyle\int \frac{\ln \cos x}{\cot x}\, dx$.

31. $\displaystyle\int \cot^3 4x \csc 4x\, dx$.

32. $\displaystyle\int \tan^3 ax \sec ax\, dx$.

80. Integrals of Trigonometric Functions.

The formulas (4) − (9), Section 77, are the inverses of the corresponding differentiation formulas. The formulas (17) − (20) cannot be directly identified with the derivatives of known functions. However, such forms can be changed so that they may be recognized as derivatives of known functions. These four formulas are obtained as follows:

$$(17) \quad \int \tan u \, du = -\int \frac{-\sin u}{\cos u} \, du = -\ln \cos u = \ln \sec u + C.$$

$$(18) \quad \int \cot u \, du = \int \frac{\cos u}{\sin u} \, du = \ln \sin u + C.$$

$$(19) \quad \int \sec u \, du = \int \frac{\sec^2 u + \sec u \tan u}{\sec u + \tan u} \, du = \ln (\sec u + \tan u) + C.$$

$$(20)\bullet \quad \int \csc u \, du = \int \frac{\csc^2 u - \csc u \cot u}{\csc u - \cot u} \, du = \ln (\csc u - \cot u) + C.$$

Integrals of Sines and Cosines. An integrand which assumes the form

$$\sin^m u \cos^n u,$$

where m and n are positive integers, is treated under two cases:

Case 1, if one exponent is odd, the integrand is transformed by the elementary relation

$$\sin^2 u + \cos^2 u = 1.$$

In illustration,

$$\int \cos^3 ax \, dx = \frac{1}{a} \int (1 - \sin^2 ax) \, a \cos ax \, dx$$

$$= \frac{1}{a} \left(\sin ax - \frac{\sin^3 ax}{3} \right) + C.$$

$$\int \sin^3 ax \cos^2 ax \, dx = -\frac{1}{a} \int (\cos^2 ax - \cos^4 ax)(- a \sin ax) \, dx$$

$$= \frac{\cos^5 ax}{5a} - \frac{\cos^3 ax}{3a} + C.$$

$$\int \sin^3 ax \cos^3 ax \, dx = \frac{\sin^4 ax}{4a} - \frac{\sin^6 ax}{6a} + C,$$

or

$$= \frac{\cos^6 ax}{6a} - \frac{\cos^4 ax}{4a} + C.$$

Case 2, if one exponent is even and the other is also even, or zero, the integrand is transformed by one of the multiple angle relations

$$2 \sin^2 u = 1 - \cos 2u,$$
$$2 \cos^2 u = 1 + \cos 2u,$$
$$\sin 2u = 2 \sin u \cos u.$$

In illustration,

$$\int \cos^2 ax \, dx = \frac{1}{2} \int (1 + \cos 2ax) \, dx = \frac{x}{2} + \frac{\sin 2ax}{4a} + C.$$

$$\int \sin^2 ax \cos^2 ax \, dx = \frac{1}{4} \int \sin^2 2ax \, dx = \frac{1}{8} \int (1 - \cos 4ax) \, dx$$

$$= \frac{x}{8} - \frac{\sin 4ax}{32a} + C.$$

Integrals of Tangents and Secants. An integrand which assumes the form

$$\tan^m u \sec^n u,$$

where m and n are positive integers may be transformed by the elementary relation

$$\tan^2 u + 1 = \sec^2 u.$$

In illustration,

$$\int \tan^4 ax \, dx = \int \tan^2 ax \, (\sec^2 ax - 1) \, dx$$

$$= \frac{\tan^3 ax}{3a} - \int (\sec^2 ax - 1) \, dx.$$

$$= \frac{\tan^3 ax}{3a} - \frac{\tan ax}{a} + x + C.$$

$$\int \tan^2 ax \sec^4 ax \, dx = \int (\tan^2 ax \sec^2 ax + \tan^4 ax \sec^2 ax) \, dx$$

$$= \frac{\tan^3 ax}{3a} + \frac{\tan^5 ax}{5a} + C.$$

The treatment of an integrand which assumes the form

$$\cot^m u \csc^n u,$$

is similar to that of tangents and secants except that one uses the relation

$$\cot^2 u + 1 = \csc^2 u.$$

In the integration of trigonometric functions the formulas of trigonometry should be used freely. More often than not, some change in the

function is necessary before an integration formula can be applied, as the above illustrations show. It frequently occurs that different trigonometric changes yield different forms of the result. Such forms usually can be shown to be equivalent by including in the constant of integration certain numerical values. Thus,

$$\int \frac{\cos^2 x - \sin^2 x}{\sin x \cos x}\, dx = 2\int \frac{\cos 2x}{\sin 2x}\, dx = 2\int \cot 2x\, dx$$

$$= \ln \sin 2x + C,$$

or

$$= \int (\cot x - \tan x)\, dx$$

$$= \ln \sin x - \ln \sec x + C.$$

Transforming the latter result,

$$\ln \sin x - \ln \sec x = \ln \frac{\sin x}{\sec x} = \ln (\sin x \cos x) = \ln \frac{\sin 2x}{2}$$

$$= \ln \sin 2x - \ln 2.$$

Hence, when the constant term $-\ln 2$ is combined with the constant C, the results are the same.

Exercise 54

GROUP A

Evaluate each of the following integrals.

1. $\int \sin^3 x\, dx.$

2. $\int \cos^3 2x \sin 2x\, dx.$

3. $\int \sin^2 2x\, dx.$

4. $\int \sec 3x \tan 3x\, dx.$

5. $\int \sin^3 x \cos^3 x\, dx.$

6. $\int \cot^4 x\, dx.$

7. $\int \left(\sin \frac{x}{2} - \cos \frac{x}{2} \right)^2 dx.$

8. $\int \sin^2 x \cos^3 x\, dx.$

9. $\int \sin^2 \frac{x}{2} \cos^2 \frac{x}{2}\, dx.$

10. $\int \sin^2 \frac{2}{3}x\, dx.$

11. $\int \sin^4 x\, dx.$

12. $\int \cos^4 2x\, dx.$

13. $\int \sqrt{1 + \cos 2x}\, dx.$

14. $\int \sqrt{1 - \cos \frac{2}{3}x}\, dx.$

15. $\int \frac{dx}{\sqrt{\cos 3x + 1}}.$

16. $\int \frac{dx}{\sqrt{1 - \cos 4x}}.$

17. $\int \frac{dx}{\sin x}.$

18. $\int \frac{dx}{\cos x}.$

19. $\int \cos^4 x \sin^3 x\, dx.$

20. $\int \sin^5 3x\, dx.$

GROUP B

Evaluate each of the following integrals.

21. $\int \frac{\cos^3 x}{\sin^4 x} dx.$

22. $\int \tan^2 2x \, dx.$

23. $\int \cot^3 x \, dx.$

24. $\int \tan^6 x \, dx.$

25. $\int \sin^7 x \, dx.$

26. $\int \tan^5 x \, dx.$

27. $\int \tan^4 x \sec^4 x \, dx.$

28. $\int \sec^6 x \, dx.$

29. $\int \sin^4 x \cos^2 x \, dx.$

30. $\int \sin^6 ax \, dx.$

31. $\int \sin^4 x \cos^4 x \, dx.$

32. $\int \sin 3x \cos 2x \, dx.$

33. $\int e^x \sin e^x \, dx.$

34. $\int e^{\sin^2 x} \sin 2x \, dx.$

35. $\int \frac{1}{x^2} e^{1/x} \, dx.$

36. $\int (3e^{2x} + 2x^{3e}) \, dx.$

37. $\int 2^{4x^2} x \, dx.$

38. $\int e^x a^x \, dx.$

39. $\int \frac{\arctan x}{1 + x^2} \, dx.$

40. $\int e^{\arctan x} \frac{dx}{1 + x^2}.$

81. Integration by Substitution.

Many integrands can be integrated by the introduction of a new variable of integration. If the original variable is x, the substitution of a new variable z is brought about by assuming a relation

$$x = f(z).$$

It is highly important to remember that not only is x replaced by the function of z, but also, that dx is obtained from that function and substituted. When the new function and its differential

$$dx = f'(z) \, dz$$

are substituted in an expression and the integration carried out with respect to z, the reverse substitution gives the result in terms of the original variable x. This process is called *integration by substitution*.

No general directions can be given as to the nature of a substitution. Its purpose is to bring the integrand into a form which will permit the application of one of the formulas of integration. Frequently, substitution

of a new variable is used to remove radicals in the integrand. Skill in the use of substitutions comes with practice only.

In general, substitutions may be considered of two types, *algebraic* and *trigonometric*.

Algebraic Substitutions. The substitution which is to be made in any particular integrand is to be determined by inspection. Consider the following integrations in which the substitutions are made for the purpose of eliminating the radicals.

$$\int \frac{\sqrt{x}+1}{\sqrt{x^3}+2x+2\sqrt{x}}\, dx = 2 \int \frac{z+1}{z^2+2z+2}\, dz,$$

·where $x = z^2$ and $dx = 2z\, dz$. Integrating and making the reverse substitution, we have

$$\ln\,(z^2+2z+2)+C = \ln\,(x+2\sqrt{x}+2)+C.$$

$$\int \frac{\sqrt{3x+1}}{3x+3}\, dx = \frac{2}{3} \int \frac{z^2\, dz}{z^2+2} = \frac{2}{3} \int \left(1 - \frac{2}{z^2+2}\right) dz,$$

where $z^2 = 3x + 1$ and $dx = \frac{2}{3}z\, dz$. Integrating and making the reverse substitution, we have

$$\frac{2}{3}z - \frac{4}{3\sqrt{2}} \arctan \frac{z}{\sqrt{2}} + C = \frac{2}{3}\sqrt{3x+1} - \frac{2\sqrt{2}}{3} \arctan \sqrt{\frac{3x+1}{2}} + C.$$

In the following integrand there is no radical and yet it cannot·be integrated by any of the standard formulas. However, the substitution used brings it to a form readily integrable, since the denominator is reduced to a monomial.

$$\int \frac{x^2\, dx}{(x+2)^2} = \int \frac{z^2 - 4z + 4}{z^2}\, dz = \int \left(1 - \frac{4}{z} + \frac{4}{z^2}\right) dz,$$

where $x = z - 2$ and $dx = dz$. Carrying out the integration and the reverse substitution,

$$z - 4 \ln z - 4z^{-1} + C = x - 4 \ln\,(x+2) - \frac{4}{x+2} + C,$$

in which the constant of integration includes the 2 omitted in the result.

Trigonometric Substitutions. If a change of variable of integration includes a trigonometric function, the substitution is known as a *trigonometric substitution*. Such substitutions have been found useful when the integrand contains certain special forms. The forms and their

substitutions are as follows:

$$\text{For } a^2 - x^2 \quad \text{use } x = a \sin \theta.$$

$$\text{For } a^2 + x^2 \quad \text{use } x = a \tan \theta.$$

$$\text{For } x^2 - a^2 \quad \text{use } x = a \sec \theta.$$

In justification of the use of these substitutions for the removal of radicals, we write the following expressions:

$$\sqrt{a^2 - x^2} = \sqrt{a^2(1 - \sin^2 \theta)} = a \cos \theta.$$

$$\sqrt{a^2 + x^2} = \sqrt{a^2(1 + \tan^2 \theta)} = a \sec \theta.$$

$$\sqrt{x^2 - a^2} = \sqrt{a^2(\sec^2 \theta - 1)} = a \tan \theta.$$

However, these substitutions are frequently used for the transformation of an integrand when there is no radical involved. Moreover, these combinations by no means exhaust the possibilities of trigonometric substitutions.

The following three integrations will serve to illustrate the use of trigonometric substitutions.

$$\int \sqrt{a^2 - x^2} \, dx = a^2 \int \cos^2 \theta \, d\theta = \frac{a^2}{2} \int (1 + \cos 2\theta) \, d\theta$$

$$= \frac{a^2}{4} (2\theta + \sin 2\theta) + C$$

$$= \frac{1}{2} (a^2 \arcsin \frac{x}{a} + x \sqrt{a^2 - x^2}) + C.$$

The change of variable is effected by letting $x = a \sin \theta$. Hence, $\theta = \arcsin \frac{x}{a}$ and $\sin 2\theta = 2 \sin \theta \cos \theta$. From Figure 65, $\cos \theta = \frac{\sqrt{a^2 - x^2}}{a}$.

$$\int \frac{x^3 \, dx}{(x^2 + 4)^{3/2}} = 2 \int \frac{\tan^3 \theta}{\sec \theta} \, d\theta = 2 \int \frac{\tan \theta \, (\sec^2 \theta - 1)}{\sec \theta} \, d\theta$$

$$= 2 \int \tan \theta \, (\sec \theta - \cos \theta) \, d\theta$$

$$= 2 \int (\tan \theta \sec \theta - \sin \theta) \, d\theta$$

$$= 2(\sec \theta + \cos \theta) + C,$$

in which $x = 2 \tan \theta$. By means of Figure 66, the reverse substitution is made, giving

$$\sqrt{x^2 + 4} + \frac{4}{\sqrt{x^2 + 4}} + C = \frac{x^2 + 8}{\sqrt{x^2 + 4}} + C.$$
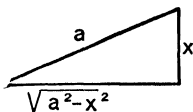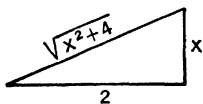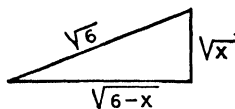
FIG. 65   FIG. 66   FIG. 67

To evaluate the following integral, we let

$$x = 6 \sin^2 \theta.$$

$$\int \frac{dx}{x\sqrt{6x - x^2}} = \frac{1}{3}\int \frac{d\theta}{\sin^2 \theta} = \frac{1}{3}\int \csc^2 \theta \, d\theta = -\frac{1}{3}\cot \theta + C.$$

Making the reverse substitution, by means of Figure 67, we have

$$-\frac{1}{3}\frac{\sqrt{6 - x}}{\sqrt{x}} = -\frac{\sqrt{6x - x^2}}{3x} + C.$$

Exercise 55

GROUP A

Evaluate each of the following integrals.

1. $\displaystyle\int x\sqrt{x + 1}\, dx.$

2. $\displaystyle\int \sqrt{x + 1}\, dx.$

3. $\displaystyle\int \frac{dx}{x\sqrt{x - 4}}.$

4. $\displaystyle\int \frac{dx}{\sqrt[3]{x + 1}}.$

5. $\displaystyle\int \frac{x}{\sqrt[3]{x + 1}}\, dx.$

6. $\displaystyle\int \frac{dx}{\sqrt{4 - x^2}}.$

7. $\displaystyle\int \frac{dx}{(4 - x^2)^{3/2}}.$

8. $\displaystyle\int \frac{x^3}{x^2 + 4}\, dx.$

9. $\displaystyle\int \frac{x^2}{x^2 + 4}\, dx.$

10. $\displaystyle\int \frac{dx}{\sqrt{x^2 + 4}}.$

11. $\displaystyle\int \frac{dx}{\sqrt{x + 3}}.$

12. $\displaystyle\int \frac{x + 2}{\sqrt{x + 3}}\, dx.$

13. $\displaystyle\int \frac{x^2}{(x + 1)^3}\, dx.$

14. $\displaystyle\int \frac{dx}{2 + \sqrt{x + 1}}.$

15. $\displaystyle\int \frac{x}{3 + \sqrt{x}}\, dx.$

16. $\displaystyle\int \frac{x}{\sqrt{x^2 + 4}}\, dx.$

17. $\displaystyle\int \frac{dx}{x\sqrt{x^2 - a^2}}.$

18. $\displaystyle\int \frac{x^2}{(x^2 + 4)^2}\, dx.$

19. $\displaystyle\int \frac{dx}{(x^2 - 9)^{3/2}}.$

20. $\displaystyle\int \frac{dx}{x^2\sqrt{a^2 - x^2}}.$

GROUP B

Evaluate each of the following integrals.

21. $\displaystyle\int \sqrt{1 + \sqrt{x}}\, dx.$

22. $\displaystyle\int \frac{x^3}{x^2 + 4}\, dx.$

23. $\displaystyle\int \frac{\sin \sqrt{x}}{\sqrt{x}}\, dx.$

24. $\displaystyle\int \frac{x^2}{(x^2 + 9)^{5/2}}\, dx.$

25. $\displaystyle\int \frac{\sqrt{x^3 - 4}}{x}\, dx.$

26. $\displaystyle\int \frac{(4x^2 - 9)^{3/2}}{x^6}\, dx.$

27. $\displaystyle\int x^5 (x^3 - 8)^{2/3}\, dx$

28. $\displaystyle\int \frac{dx}{x^2 \sqrt{9 + x^2}}.$

29. $\displaystyle\int \sqrt{\sqrt{x} - 1}\, dx.$

30. $\displaystyle\int \frac{x^2}{(4 - x^2)^{7/2}}\, dx.$

31. $\displaystyle\int \frac{dx}{\sqrt{1 - \sqrt{x}}}.$

32. $\displaystyle\int \frac{x^3}{(x^2 - 4)^3}\, dx.$

33. $\displaystyle\int \frac{dx}{x(x + 6)}.$

34. $\displaystyle\int \frac{dx}{\sqrt{e^{2x} - 9}}.$

35. $\displaystyle\int \frac{\sqrt{x}}{(9x + 4)^2}\, dx.$

36. $\displaystyle\int \frac{x^3}{(1 + 4x)^{5/2}}\, dx.$

GROUP C

Derive formulas for the evaluation of each of the following integrals.

37. $\displaystyle\int \frac{du}{\sqrt{u^2 + a^2}} = \ln (u + \sqrt{u^2 + a^2}) + C.$

38. $\displaystyle\int \frac{du}{\sqrt{u^2 - a^2}} = \ln (u + \sqrt{u^2 - a^2}) + C.$

39. $\displaystyle\int \frac{du}{u\sqrt{2au - u^2}} = -\frac{\sqrt{2au - u^2}}{au} + C.$

40. $\displaystyle\int \frac{du}{u\sqrt{2au + u^2}} = -\frac{\sqrt{2au + u^2}}{au} + C.$

41. $\displaystyle\int \frac{du}{u\sqrt{u^2 - 2au}} = \frac{\sqrt{u^2 - 2au}}{au} + C.$

82. Integration by Parts.

The formula for the differential of the product of two functions is

$$d(uv) = u\, dv + v\, du.$$

Integrating both sides of the equation,

$$uv = \int u\, dv + \int v\, du.$$

Solving for one of the integrals, we have an important formula of integration,

(21) $$\int u\ dv = uv - \int v\ du.$$

Integration by means of this formula is called *integration by parts*. The application of this formula succeeds in many cases when the methods presented in previous sections fail. The success of the application of the method of integration by parts usually depends on the ability to choose u and dv so that the integral, $\int v\ du$, can be evaluated.

The following evaluations illustrate the application of integration by parts.

$$\int x \cos x\ dx = x \sin x - \int \sin x\ dx$$
$$= x \sin x + \cos x + C,$$

where

$$u = x, \quad dv = \cos x\ dx,$$

and hence,

$$du = dx, \quad v = \sin x.$$

In this same problem, it may be pointed out, that had we let

$$u = \cos x, \quad dv = x\ dx,$$

then

$$du = -\sin x\ dx, \quad v = \frac{x^2}{2},$$

and

$$\int x \cos x\ dx = \frac{x^2}{2} \cos x + \frac{1}{2} \int x^2 \sin x\ dx,$$

in which the latter integrand is more difficult to evaluate than is the given one.

$$\int x^3 e^{x^2}\ dx = \tfrac{1}{2} x^2 e^{x^2} - \int x e^{x^2}\ dx$$
$$= \tfrac{1}{2}(x^2 e^{x^2} - e^{x^2}) + C,$$

where

$$u = x^2, \quad dv = x e^{x^2}\ dx$$

and

$$du = 2x\ dx, \quad v = \tfrac{1}{2} e^{x^2}.$$

Sometimes the method of integration by parts should be applied two or more times before an integrand is obtained which can be evaluated, as illustrated by the following:

$$\int x^2 \sin x \, dx = -x^2 \cos x + 2 \int x \cos x \, dx$$

$$= -x^2 \cos x + 2(x \sin x + \cos x) + C.$$

$$\int e^x \sin x \, dx = -e^x \cos x + \int e^x \cos x \, dx,$$

where

$$u = e^x, \quad dv = \sin x \, dx.$$

Upon a second application,

$$\int e^x \sin x \, dx = -e^x \cos x + e^x \sin x - \int e^x \sin x \, dx,$$

where

$$u = e^x, \quad dv = \cos x \, dx.$$

Transposing the integral in the second member, we have

$$\int e^x \sin x \, dx = \tfrac{1}{2} e^x (\sin x - \cos x) + C.$$

Certain integrands, such as

$$\ln x, \ \arctan x, \ \arcsin x, \ \text{etc.},$$

cannot be integrated directly. But since they are differentiable, integration by parts serves to evaluate them. For the first,

$$\int \ln x \, dx = x \ln x - \int dx = x \ln x - x + C.$$

Exercise 56

Evaluate each of the following integrals.

1. $\displaystyle\int x e^{2x} \, dx.$

2. $\displaystyle\int x \ln x \, dx.$

3. $\displaystyle\int x \sin x \, dx.$

4. $\displaystyle\int \arctan x \, dx.$

5. $\displaystyle\int x^2 e^{3x} \, dx.$

6. $\displaystyle\int x^2 \sin 2x \, dx.$

7. $\int x^2 \ln x \, dx.$

8. $\int x \sin^2 3x \, dx.$

9. $\int x \arctan x \, dx.$

10. $\int x \arcsin x \, dx.$

11. $\int x^2 \ln x^2 \, dx.$

12. $\int x \arctan x^2 \, dx.$

13. $\int x \arctan^2 x \, dx.$

14. $\int \sin x \ln \cos x \, dx.$

15. $\int e^x \cos x \, dx.$

16. $\int x^3 \sqrt{4 - x^2} \, dx.$

17. $\int x^3 e^{-x^2} \, dx.$

18. $\int x^2 \arcsin x \, dx.$

GROUP B

Evaluate each of the following integrals.

19. $\int \dfrac{dx}{x \ln x}.$

20. $\int \sqrt{\dfrac{a + x}{a - x}} \, dx.$

21. $\int \dfrac{e^{\sqrt{x}} + 1}{\sqrt{x}} \, dx.$

22. $\int e^{\tan x} \sec^2 x \, dx.$

23. $\int (e^{ax} - e^{-ax})^2 \, dx.$

24. $\int \dfrac{dx}{(x + 1) + \sqrt{x + 1}}.$

25. $\int \dfrac{\sqrt{4 - x^2}}{x^4} \, dx.$

26. $\int \dfrac{\arctan x}{x^2} \, dx.$

27. $\int \dfrac{\sin x}{e^x} \, dx.$

28. $\int \dfrac{xe^x}{(1 + x)^2} \, dx.$

29. $\int \dfrac{x^2}{e^x} \, dx.$

30. $\int \arctan \dfrac{2}{x} \, dx.$

31. $\int \dfrac{\arctan \sqrt{x}}{x^2} \, dx.$

32. $\int \ln (1 - \sqrt{x}) \, dx.$

33. $\int e^{-x} \sin 3x \, dx.$

34. $\int \sec^3 x \, dx.$

35. $\int \csc^3 x \, dx.$

36. $\int \dfrac{xe^{-x}}{(1 - x)^2} \, dx.$

83. Integrals of Rational Fractions.

The quotient of two polynomials is called a *rational fraction*. While the integration of special cases of rational fractions has been found possible by the methods of previous sections, these methods do not apply for all cases.

Regardless of later considerations, the first step in the integration of a rational fraction is to *perform the indicated division until the numerator is of lower degree than the denominator.*

Partial Fractions. The *partial fractions* of a rational fraction are those fractions whose sum is the original fraction and whose denominators are the real factors of the denominator of the original fraction. The separation of a fraction into its partial fractions is a problem of algebra which is considered under three headings: The denominator can be factored into real linear factors, no one of which is repeated. The denominator can be factored into real linear factors, at least one of which is repeated. And the denominator has at least one quadratic factor.

Distinct Linear Factors. A fraction whose numerator is of lower degree than the denominator and whose denominator can be factored into several distinct linear factors can be separated into the same number of partial fractions. For example, for the fraction

$$\frac{x^2 - 3x + 5}{x^3 - 2x^2 - 5x + 6} = \frac{x^2 - 3x + 5}{(x - 1)(x + 2)(x - 3)},$$

we assume that

$$\frac{x^2 - 3x + 5}{x^3 - 2x^2 - 5x + 6} = \frac{A}{x - 1} + \frac{B}{x + 2} + \frac{C}{x - 3},$$

in which A, B and C are constants to be determined. Clearing of fractions,

$$x^2 - 3x + 5 = A(x + 2)(x - 3) + B(x - 1)(x - 3) + C(x - 1)(x + 2).$$

Since we have assumed that the original fraction is equivalent to the sum of the partial fractions, the latter equation must hold for all values of x. Consequently, we may choose the most convenient values of x which will enable us to find the values of the unknown constants. Choosing those values of x which will make each factor zero in turn,

$$x = 1, \qquad 2A = -1,$$
$$x = -2, \qquad B = 1,$$
$$x = 3, \qquad 2C = 1.$$

Using these results,

$$\int \frac{x^2 - 3x + 5}{x^3 - 2x^2 - 5x + 6}\,dx = -\frac{1}{2}\int \frac{dx}{x - 1} + \int \frac{dx}{x + 2} + \frac{1}{2}\int \frac{dx}{x - 3}$$

$$= -\ln\sqrt{x - 1} + \ln(x + 2) + \ln\sqrt{x - 3} + C,$$

$$= \ln \frac{(x + 2)\sqrt{x - 3}}{\sqrt{x - 1}} + C.$$

Repeated Linear Factors. If the denominator of a rational fraction contains a repeated linear factor, the first method of separation into partial fractions fails, since there would be several partial fractions having the same denominator which could be combined into a single one. The following problem illustrates the method of procedure.

For the given fraction, we assume the constants A, B, C and D,

$$\frac{x^2 + 1}{x(x + 1)^3} = \frac{A}{x} + \frac{B}{x + 1} + \frac{C}{(x + 1)^2} + \frac{D}{(x + 1)^3}.$$

Clearing of fractions,

$$x^2 + 1 = A(x + 1)^3 + Bx(x + 1)^2 + Cx(x + 1) + Dx.$$

As in the first method,

$$x = 0, \quad A = 1. \quad x = -1, \quad D = -2.$$

This exhausts the possibilities of choosing those values of x for which a factor is zero. To determine the two remaining constants, we may use the method of equating the coefficients of like powers of x, or we may choose two convenient values of x other than those already used. Sometimes it is advantageous to combine the two methods to obtain suitable equations in terms of the unknown constants. Let us equate the coefficients of x^3 and those of x. Since the coefficient of each is zero in the left-hand member of the equation, these are the more convenient coefficients. Thus,

$$A + B = 0, \quad B = -A = -1,$$

$$3A + B + C + D = 0, \quad C = 0.$$

Using these results,

$$\int \frac{x^2 + 1}{x(x + 1)^3}\, dx = \int \frac{dx}{x} - \int \frac{dx}{x + 1} - 2 \int \frac{dx}{(x + 1)^3}$$

$$= \ln x - \ln(x + 1) + \frac{1}{(x + 1)^2} + C$$

$$= \ln \frac{x}{x + 1} + \frac{1}{(x + 1)^2} + C.$$

Quadratic Factors. Corresponding to every quadratic factor of the denominator of a rational fraction, we assume a linear numerator for the partial fraction. Such a quadratic factor is assumed not to have real linear factors, otherwise the separation into partial fractions is one of the two preceding cases.

For illustration, we choose a rational fraction whose denominator has a nonrepeated quadratic factor and assume the constants A, B and C,

$$\frac{3x^2 - 2x + 1}{x^3 - 2x^2 + 2x - 1} = \frac{A}{x - 1} + \frac{Bx + C}{x^2 - x + .1}.$$

Clearing of fractions,

$$3x^2 - 2x + 1 = A(x^2 - x + 1) + Bx(x - 1) + C(x - 1).$$

Letting $x = 1$, $x = 0$ and equating coefficients of x^2, we find ·

$$A = 2, \quad C = 1, \quad B = 1.$$

Using these results,

$$\int \frac{3x^2 - 2x + 1}{x^3 - 2x^2 + 2x - .1} \, dx = 2 \int \frac{dx}{x - 1} + \frac{1}{2} \int \frac{2x - 1}{x^2 - x + 1} \, dx +$$
$$\frac{3}{2} \int \frac{dx}{(x - \frac{1}{2})^2 + \frac{3}{4}} = \ln\left[(x - 2)^2 \sqrt{x^2 - x + 1}\right] + \sqrt{3} \arctan \frac{2x - 1}{\sqrt{3}} + C.$$

A rational fraction whose denominator has a quadratic factor repeated is treated as follows:

$$\frac{2x^4 + x^3 + 4x^2 + 1}{x(x^2 + 1)^2} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1} + \frac{Dx + E}{(x^2 + 1)^2},$$
$$2x^4 + x^3 + 4x^2 + 1 = A(x^2 + 1)^2 + Bx^2(x^2 + 1) +$$
$$Cx(x^2 + 1) + Dx^2 + Ex.$$

The constants are found by equating coefficients of constant term, x^4, x^3, x, and x^2 in the given order, giving

$$A = 1, \quad B = 1, \quad C = 1, \quad E = -1, \quad D = 1.$$

Hence,

$$\int \frac{2x^4 + x^3 + 4x^2 + 1}{x(x^2 + 1)^2} \, dx =$$

$$\int \frac{dx}{x} + \int \frac{x \, dx}{x^2 + 1} + \int \frac{dx}{x^2 + 1} + \int \frac{x \, dx}{(x^2 + 1)^2} - \int \frac{dx}{(x^2 + 1)^2}$$

$$= \ln x + \frac{1}{2} \ln (x^2 + 1) + \arctan x - \frac{1}{2} \frac{1}{x^2 + 1} -$$

$$\frac{1}{2} \arctan x - \frac{1}{2} \frac{x}{x^2 + 1} + C$$

$$= \ln (x\sqrt{x^2 + 1}) + \frac{1}{2} \arctan x - \frac{x + 1}{2(x^2 + 1)} + C,$$

where the last integral was evaluated by means of trigonometric substitution.

Exercise 57

GROUP A

Evaluate each of the following integrals.

1. $\int \dfrac{x+4}{x^2-2x}\,dx.$

2. $\int \dfrac{x+8}{x^2+x-6}\,dx.$

3. $\int \dfrac{x-3}{3x^2-2x-1}\,dx.$

4. $\int \dfrac{x^2+4x-4}{x^3-4x}\,dx.$

5. $\int \dfrac{x^2+x+4}{(x+3)(x^2-4)}\,dx.$

6. $\int \dfrac{x^2+6x-1}{6x^3+x^2-x}\,dx.$

7. $\int \dfrac{8x+24}{x^3+4x^2}\,dx.$

8. $\int \dfrac{2x-5}{x(x-1)^2}\,dx.$

9. $\int \dfrac{dx}{x^4+x^3}.$

10. $\int \dfrac{dx}{x^3-x^2}.$

11. $\int \dfrac{x^2+x+4}{x^3+x}\,dx.$

12. $\int \dfrac{x^2+2}{x^2-4x+5}\,dx.$

13. $\int \dfrac{\sqrt{x}+1}{2x(\sqrt{x}-1)}\,dx.$

14. $\int \dfrac{x^4}{x^3+1}\,dx$

GROUP B

Evaluate each of the following integrals.

15. $\int \dfrac{x^2}{x^2+2x+10}\,dx.$

16. $\int \dfrac{x+1}{\sqrt{3-2x-x^2}}\,dx.$

17. $\int \dfrac{x+1}{\sqrt{3+2x-x^2}}\,dx.$

18. $\int \dfrac{\cos x}{\sin x+\sin^2 x}\,dx.$

19. $\int \dfrac{\sec^2 x}{\tan^3 x+4\tan x}\,dx.$

20. $\int \dfrac{\tan x}{1-\cos x}\,dx.$

21. $\int \dfrac{dx}{x^2\sqrt{1-x}}.$

22. $\int \dfrac{5x^2-x+4}{x^3-x^2+2x}\,dx.$

23. $\int \dfrac{x^2+2x-4}{(x-1)^2(x-2)}\,dx.$

24. $\int \dfrac{x^3+x^2-3x-2}{(x^2+2)(x-1)^2}\,dx.$

25. $\int \dfrac{dx}{x(x^2-2x+2)^2}.$

26. $\int \dfrac{x\ln x}{(1+x^2)^2}\,dx.$

27. $\int \dfrac{x\ln x}{(1+x^2)^3}\,dx.$

28. $\int \dfrac{\cos^2 x}{\cos x+1}\,dx.$

GROUP C

Derive formulas for the evaluation of the following integrals.

29. $\displaystyle\int \frac{du}{u^2 - a^2} = \frac{1}{2a} \ln \frac{u - a}{u + a} + C$, where $|u| > |a|$.

30. $\displaystyle\int \frac{du}{a^2 - u^2} = \frac{1}{2a} \ln \frac{a - u}{a + u} + C$, where $|u| < |a|$.

31. $\displaystyle\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a} \operatorname{arcsec} \frac{u}{a} + C$, where $|u| > |a|$.

84. Definite Integrals.

In Chapter VI the definite integral is shown to be independent of the constant of integration. Thus

$$\int_a^b f(x)\, dx = F(x) \Big]_a^b = F(b) - F(a),$$

where $\dfrac{d}{dx} F(x) = f(x)$. Hence, a definite integral is a function of its limits. In addition, it was assumed that the function $f(x)$ is continuous and single-valued in the interval from $x = a$ to $x = b$ and that the function is integrable.

Change of Limits. When a definite integral is evaluated by a change of the variable of integration, the *limits of the integral should also be changed to correspond with the change of variable.* This change of limits obviates the necessity of making a reverse substitution of variable. For example,

$$\int_{-2}^{2} \frac{dx}{\sqrt{x^2 + 4}} = \int_{-\pi/4}^{\pi/4} \sec \theta\, d\theta = \ln (\sec \theta + \tan \theta) \Big]_{-\pi/4}^{\pi/4}$$

$$= \ln (\sqrt{2} + 1) - \ln (\sqrt{2} - 1) = \ln (3 + 2\sqrt{2}).$$

Also,

$$\int_4^9 \frac{\sqrt{x}}{\sqrt{x} + 1}\, dx = 2 \int_2^3 \frac{z^2}{z + 1}\, dz = z^2 - 2z + 2 \ln (z + 1) \Big]_2^3$$

$$= 3 - \ln \frac{16}{9}.$$

Exercise 58

GROUP A

Evaluate each of the following definite integrals.

1. $\int_0^{3/4} x\sqrt{x^2+1}\,dx.$

2. $\int_{-3/4}^0 \sqrt{x+1}\,dx.$

3. $\int_{-1}^1 \frac{dx}{\sqrt{4-x^2}}.$

4. $\int_{-2}^2 \frac{dx}{x^2+4}.$

5. $\int_0^3 x\sqrt{x+1}\,dr.$

6. $\int_0^2 (4-x^2)^{3/2}\,dx.$

7. $\int_0^{\pi/2} \sin^2 x \cos x\,dx.$

8. $\int_{1/2}^1 e^{2x-1}\,dx.$

9. $\int_{5/4}^{5/3} \frac{x}{\sqrt{x^2-1}}\,dr.$

10. $\int_0^1 x^2 e^{x^3}\,dx.$

11. $\int_3^5 \frac{dx}{\sqrt{x^2-9}}.$

12. $\int_2^5 \frac{dx}{2x^2-5x+3}.$

13. $\int_0^5 \frac{dx}{\sqrt{9-x}}$

14. $\int_{-1/2}^1 \arcsin x\,dx.$

15. $\int_1^{\sqrt{3}} \frac{dr}{(x^2+1)^{5/2}}.$

16. $\int_{3\sqrt{2}}^6 \frac{dx}{x^2\sqrt{x^2-9}}.$

17. $\int_1^e \ln x\,dx.$

18. $\int_0^1 \arctan x\,dx.$

19. $\int_0^{\pi/4} x \sin x\,dx.$

20. $\int_0^{\pi/8} \cos 2x \sin 4x\,dx.$

21. $\int_0^{\pi} x \sin^2 x\,dr.$

22. $\int_2^4 x^3\sqrt{x^2-4}\,dx.$

23. $\int_1^2 \sqrt{2x-x^2}\,dx.$

24. $\int_{-1/2}^{1/2} \frac{dr}{\sqrt{7+4x-4x^2}}.$

GROUP B

25. Find the area bounded by $xy = 12$ and $x + y = 8$.
26. Find the area of a circle of radius a by integration.
27. Find the area of the ellipse $b^2x^2 + a^2y^2 = a^2b^2$.
28. Find the area of the loop of the curve $y^2 = x^2(2 - x)$.
29. Find the area bounded by $y(x^2 + 4) = 8$, the x-axis, $x + 2 = 0$ and $x - 2 = 0$.
30. Find the area under the curve $y^2(4 - x^2) = 4$ from $x = -2$ to $x = 2$.
31. Find the area under the curve $y = \arcsin x$ from $x = 0$ to $x = \frac{1}{2}$.
32. Find the area under the curve $y = \arctan x$ from $x = 0$ to $x = 1$.
33. Find the area in the first quadrant bounded by $y = \cos x$, $y = \sin 2x$ and the y-axis.
34. Find the area under the curve $y^2(x^2 + 4) = 4$ and above the x-axis between $x = -2$ and $x = 2$.

35. Find the volume generated by rotating the area under one arch of $y = \sin x$ about the x-axis.

36. Find the volume generated by rotating the area under one arch of $y = \cos 2x$ about the x-axis.

37. Find the volume generated by rotating the area under the curve $y(x^2 + x) = 1$ from $x = 1$ to $x = 2$ about the x-axis.

38. Find the volume generated by rotating the area in the first quadrant bounded by $y = \cos x$, $y = \sin 2x$ and the y-axis about the x-axis.

39. Find the volume generated by rotating the area under one arch of the cycloid $x = a(\theta - \sin \theta)$, $y = a(1 - \cos \theta)$ about the x-axis.

40. Draw the curve $y(x^2 + 4) = 4x$, find the coordinates of the maximum, minimum and inflection points, find the slope at the origin and find the area in the first quadrant bounded by the curve, the x-axis, the maximum ordinate and the ordinate of the inflection point.

GROUP C

Differentiate each of the following functions.

41. $e^{\arctan x^2}$.

42. $\ln (\ln \tan x)$.

43. $a^{\ln \sin 3x}$.

44. $x^{\cos x}$.

Draw each of the following curves.

45. $y = 3 \ln \dfrac{x}{2}$.

46. $y = x^3 e^{-x}$.

47. $y = e^{1 + (1/x)}$.

48. $y = \frac{1}{2}(e^{x/2} + e^{-x/2})$.

49. A ladder 80 ft. long rests against the vertical wall of a house. A fence of greatest height is to be built 10 ft. from the house. Find the height of the fence so that the ladder just clears it.

50. Draw the curve $y = x^2 - 4 \arctan x$ from $x = 0$ to $x = \sqrt{3}$. Find the coordinates of the minimum point and find the area from that ordinate between the curve and the x-axis to $x = \sqrt{3}$.

51. Find the volume generated by rotating the area bounded by $y = e^x$, the x-axis, $x = -2$ and the y-axis about the line $y + 1 = 0$.

52. Find the area in the first quadrant under $y = \dfrac{e^x}{e^{2x} - 1}$ between $x = 1$ and $x = 2$.

53. The slope of a curve at any point is y/x^2. If the curve passes through the point $(1, e)$, find its equation.

54. Find the volume generated by rotating the area bounded by $y = \arctan x$, the x-axis and the line $x = 1$ about the y-axis.

55. A particle is moved along the x-axis by the action of a force numerically equal to $x/\sqrt{25 - x^2}$, where x is measured in inches. Find the work done by the force in moving the particle from the origin to $x = 4$ ins.

85. Improper Integrals.

By the fundamental theorem of integral calculus

$$\int_a^b f(x)\, dx = \lim_{\Delta x \to 0} \sum_{i=1}^{i=n} f(x_i)\, \Delta x,$$

where

$$x_1 = a, \quad x_n = b - \Delta x \quad \text{and} \quad (b - a)/n = \Delta x.$$

Thus we have the geometric interpretation of the definite integral as the limit of the sum of the elements of area where that area is *bounded* by the curve $y = f(x)$, $x = a$, $x = b$ and the x-axis. This limit exists provided that $f(x)$ is a continuous single-valued function in the interval

$$a \leqq x \leqq b,$$

and that the function is integrable. On the other hand, this limit may or may not exist if $f(x)$ possesses a discontinuity at any point in that interval.

If the curve $y = f(x)$ has a horizontal asymptote, the area is not bounded in the ordinary sense. Such a curve is drawn in Figure 68.

If $f(x)$ is infinite for any value of x in the interval, including the end points, the curve $y = f(x)$ has a vertical asymptote and, again, the area is not bounded in the ordinary sense. Such a curve is drawn in Figure 69.

A definite integral whose integrand becomes infinite for one or more values of the variable of integration in the interval of that integration or which has an infinite limit is called an *improper integral*. The evaluation of an improper integral requires special attention.

Infinite Limits. An improper integral whose upper or lower limit is infinite, must be evaluated by the *limit process*. Whether or not this limit exists, depends on the nature of the integrand.

If the upper limit is infinite,

$$\int_a^\infty f(x)\, dx = \lim_{b \to \infty} \int_a^b f(x)\, dx.$$

And if the lower limit is infinite,

$$\int_{-\infty}^b f(x)\, dx = \lim_{a \to -\infty} \int_a^b f(x)\, dx.$$

The area in the first quadrant between the curve $xy = 1$ and the x-axis to the right of $x = 2$ is infinite. This is known from the evaluation

$$\int_2^\infty \frac{dx}{x} = \lim_{b \to \infty} \ln x \bigg]_2^b = \lim_{b \to \infty} (\ln b - \ln 2) = \infty.$$

The limit does not exist, and the integral has no meaning.

The area in the first quadrant between the curve $x^2y = 1$ and the x-axis to the right of $x = 2$ is represented in Figure 68. This area can be found by the evaluation

$$\int_2^\infty \frac{dx}{x^2} = \lim_{b\to\infty} \left(-\frac{1}{x}\right)\Big]_2^b = \lim_{b\to\infty} -\left(\frac{1}{b} - \frac{1}{2}\right) = \frac{1}{2},$$

since the limit does exist and the integral does have meaning.
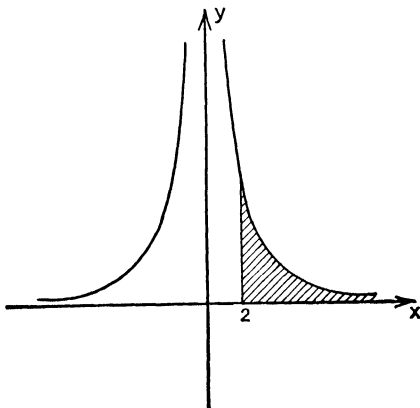
FIG. 68

Infinite Discontinuities of the Integrand. An improper integral whose integrand is discontinuous in the interval of integration, must be evaluated by the limit process.

If the integrand is infinite at the lower limit,

$$\int_a^b f(x)\, dx = \lim_{h\to 0} \int_{a+h}^b f(x)\, dx.$$

If the integrand is infinite at the upper limit,

$$\int_a^b f(x)\, dx = \lim_{h\to 0} \int_a^{b-h} f(x)\, dx.$$

And if the integrand is infinite at $x = c$, where $a < c < b$,

$$\int_a^b f(x)\, dx = \lim_{h_1\to 0} \int_a^{c-h_1} f(x)\, dx + \lim_{h_2\to 0} \int_{c+h_2}^b f(x)\, dx.$$

To find the area in the first quadrant between the curve $xy^2 = 1$, the x-axis, the y-axis and $x = 4$, we note that the y-axis is an asymptote. Hence, the following integrand is discontinuous at the first end point of

the interval of integration. The evaluation

$$\lim_{a \to 0} \int_a^4 \frac{dx}{\sqrt{x}} = \lim_{a \to 0} 2(\sqrt{x}) \Big]_a^4 = \lim_{a \to 0} (4 - 2\sqrt{a}) = 4,$$

shows that the integral has meaning, since the limit exists.

For the area in the first quadrant between the curve $x^2y = 1$, the x-axis, the y-axis and $x = 2$, we observe that the following integrand is discontinuous at $x = 0$. The curve is drawn in Figure 68. The evaluation

$$\lim_{a \to 0} \int_a^2 \frac{dx}{x^2} = \lim_{a \to 0} - \left(\frac{1}{x}\right)\Big]_a^2 = \lim_{a \to 0} - \left(\frac{1}{2} - \frac{1}{a}\right) = \infty,$$

shows that the integral has no meaning, since the limit does not exist.

For the area between the curve $y(x - 2)^2 = 1$, the x-axis, $x = 1$ and $x = 3$, it is highly important to know that the curve is discontinuous for
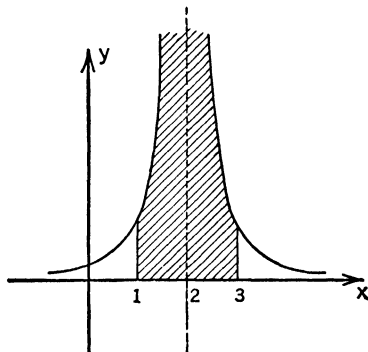
FIG. 69

$x = 2$. The curve is drawn in Figure 69. Were the integral evaluated without observing the discontinuity, the *result would be wrong*, thus

$$\int_1^3 \frac{dx}{(x - 2)^2} = -\left(\frac{1}{x - 2}\right)\Big]_1^3 = -(1 + 1) = -2.$$

In fact, the integral has no meaning. That the limit does not exist is shown by the following evaluation:

$$\lim_{h_1 \to 0} \int_1^{2 - h_1} \frac{dx}{(x - 2)^2} + \lim_{h_2 \to 0} \int_{2 + h_2}^3 \frac{dx}{(x - 2)^2} =$$

$$\lim_{h_1 \to 0} - \left(\frac{1}{x - 2}\right)\Big]_1^{2 - h_1} + \lim_{h_2 \to 0} - \left(\frac{1}{x - 2}\right)\Big]_{2 + h_2}^3 =$$

$$\lim_{h_1 \to 0} - \left(-\frac{1}{h_1} + 1\right) + \lim_{h_2 \to 0} - \left(1 - \frac{1}{h_2}\right) = \infty.$$

Exercise 59

GROUP A

Evaluate each of the following integrals, if possible.

1. $\int_1^\infty \dfrac{dx}{x^2}.$

2. $\int_0^6 \dfrac{dx}{x^3}.$

3. $\int_0^\infty x e^{x^2}\, dx.$

4. $\int_0^\infty x e^{-x^2}\, dx.$

5. $\int_0^2 \dfrac{dx}{(x-2)^2}.$

6. $\int_2^3 \dfrac{dx}{\sqrt{x-2}}.$

7. $\int_1^\infty \dfrac{x}{x^2+4}\, dx.$

8. $\int_{-1}^1 \dfrac{dx}{\sqrt[3]{x}}.$

9. $\int_0^\infty \dfrac{x}{x^4+4}\, dx.$

10. $\int_2^\infty \dfrac{dx}{x\sqrt{x^2-1}}.$

11. $\int_2^\infty \dfrac{x}{x^2-4}\, dx.$

12. $\int_{-2}^2 \dfrac{dx}{\sqrt{4-x^2}}.$

13. $\int_0^2 \dfrac{dx}{(x-1)^2}.$

14. $\int_{-3}^3 \dfrac{dx}{x^2\sqrt{9-x^2}}.$

15. $\int_0^1 \dfrac{\sqrt{1+x^2}}{x}\, dx.$

16. $\int_1^\infty \ln x\, dx.$

GROUP B

Evaluate each of the following integrals, if possible.

17. $\int_0^\infty \sin x\, dx.$

18. $\int_0^\pi \cos 2x\, dx.$

19. $\int_0^{\pi/2} \tan x\, dx.$

20. $\int_0^{\pi/2} \sec^2 x\, dx.$

21. $\int_0^1 \dfrac{e^x+1}{e^x-1}\, dx.$

22. $\int_0^3 \dfrac{x}{(x^2-1)^{4/3}}\, dx.$

23. $\int_0^4 \dfrac{1}{(3-5x)^{5/3}}\, dx.$

24. $\int_0^1 \dfrac{dx}{x(1+x^2)}.$

25. $\int_1^\infty \dfrac{dx}{x\sqrt{x+1}}.$

26. $\int_0^3 \dfrac{4}{x^2-3x+2}\, dx.$

GROUP C

Investigate the area under each of the following curves between the given limits and find that area if it exists.

27. In the first quadrant under $xy^2 = 1$ from $x = 0$ to $x = 1$ and from $x = 4$ to $x = \infty$.

28. In the first quadrant under $x^2y = 1$ from $x = 0$ to $x = 4$ and from $x = 4$ to $x = \infty$.

29. In the first quadrant under $x^3y = 1$ from $x = 0$ to $x = 2$ and from $x = 2$ to $x = \infty$.

30. The area under $x^2y^3 = 1$ from $x = -1$ to $x = 1$.

31. The area under $x^4y^3 = 1$ from $x = -1$ to $x = 1$.

32. The area between the curve $y(x^2 + 4) = 8$ and its asymptote.

33. The area in the first quadrant between the curve $y^2(1 - x^2) = 1$ and its asymptote.

34. The area in the first quadrant under $y^2(4 - x) = x^2$ from $x = 0$ to $x = 4$.

35. The area between the x-axis and the curve $y(x - 4)^3 = 8$ from $x = 1$ to $x = 4$.

36. The area under the curve $y^3(x - 1)^2 = 8x^3$ from $x = 0$ to $x = 3$.

37. The area between the curve $y(x^2 - 5x + 6) = 6$ and the x-axis from $x = 0$ to $x = 4$.

38. The area between the curve $y = \ln x$ and the x-axis from $x = 0$ to $x = 1$, it being given that $\lim\limits_{x \to 0} [x \ln x] = 0$.

39. Investigate the volume generated by rotating the area in the first quadrant under the curve $y = e^{-x}$ about the x-axis, and find that volume if it exists.

40. Investigate the volume generated by rotating the area between the curve $y = \ln x$ and the x-axis from $x = 0$ to $x = 1$ about the x-axis, and find that volume if it exists.

86. Table of Integrals.

It is unnecessary at this stage in the development of the integral calculus to go farther into the theory and methods required to evaluate integrals which are more complex than those already given. Some of the integrals which ordinarily arise in practice and which can be evaluated in terms of elementary functions have been collected in the form of a table appended to this text. This table is a very brief one. A more complete one is to be found in *Short Table of Integrals* by B. O. Peirce.

In the table appended, only those integrals appear which are not immediately reducible to a standard form. Those which are immediately reducible to a standard form appear in the first section of this chapter.

If an integral to be evaluated has the form of one given in the table, the result can be written immediately. In other cases a transformation may be required to bring the integrand into the form of that of the table.

It is to be observed that some of the formulas of the table express an integral in terms of a simpler one. Such formulas are called *reduction formulas,* which with many others are derived in more advanced calculus texts. It frequently happens that a reduction formula must be applied several times before the final integral can be evaluated.

To use a table of integrals intelligently, one must possess some ability in the technique of integration. The purpose of such tables is the saving of time and labor and not to obviate the necessity of knowing how to integrate.

To illustrate the uses to which the table may be put, the two following evaluations are made:

$$\int x\sqrt{1 - x^{2/3}}\, dx = 3\int \sin^5 \theta \cos^2 \theta\, d\theta,$$

in which the substitution $x = \sin^3 \theta$ is used. Applying the reduction formula 43 once and formula 41 twice, we have

$$\int x\sqrt{1 - x^{2/3}}\, dx = \frac{\cos \theta}{35}(15 \sin^6 \theta - 3 \sin^4 \theta - 4 \sin^2 \theta - 8),$$

$$= \frac{\sqrt{1 - x^{2/3}}}{35}(15x^2 - 3x^{4/3} - 4x^{2/3} - 8) + C.$$

$$\int \frac{\sin 2x}{\sqrt{1 + \sin x}}\, dx = 2\int \sin x\sqrt{1 - \sin x}\, dx,$$

$$= 2\int \frac{z}{\sqrt{1 + z}}\, dz,$$

in which the substitution $\sin x = z$ is used. Applying the formula 7,

$$\int \frac{\sin 2x}{\sqrt{1 + \sin x}}\, dx = \tfrac{4}{3}(z - 2)\sqrt{1 + z} = \tfrac{4}{3}(\sin x - 2)\sqrt{1 + \sin x} + C.$$

Exercise 60

Evaluate each of the following integrals by using the table of integrals.

1. $\displaystyle\int \sin \sqrt{x}\, dx = 2(\sin \sqrt{x} - \sqrt{x} \cos \sqrt{x}) + C.$

2. $\displaystyle\int \sqrt{1 + \sin 2x}\, dx.$

3. $\displaystyle\int \sqrt{\frac{1 - x}{1 + x}}\, dx = \sqrt{1 - x^2} + \arcsin x + C.$

4. $\displaystyle\int \frac{dx}{e^x + 1}.$

5. $\displaystyle\int \frac{dx}{e^x + e^{-x}} = \arctan e^x + C.$

6. $\displaystyle\int \frac{dx}{\sqrt{x + 3} - \sqrt{x + 2}}$

7. $\displaystyle\int \frac{dx}{x - \sqrt{x^2 + 1}} = -\tfrac{1}{2}(x^2 + x\sqrt{x^2 + 1}) + \ln (x + \sqrt{x^2 + 1}) + C.$

8. $\displaystyle\int \frac{dx}{\sqrt{e^x + 1}}.$

9. $\int \frac{\sqrt{1 + \ln x}}{x} dx = \frac{2}{3}(1 + \ln x)^{3/2} + C.$

10. $\int \frac{e^{2x}}{\sqrt{1 + e^x}} dx.$

11. $\int \frac{dx}{\sqrt{1 + \sqrt{1 + x}}} = \frac{4}{3}(\sqrt{1 + x} - 2)\sqrt{1 + \sqrt{1 + x}} + C.$

12. $\int \sqrt{x^2 + 2x + 2} \, dx.$

13. $\int \frac{x^3}{\sqrt{1 + x^2}} dx = \frac{1}{3}(x^2 - 2)\sqrt{x^2 + 1} + C.$

14. $\int \frac{\sin 2x}{\sqrt{1 + \sin^2 x}} dx.$

15. $\int x^4 \ln x \, dx = \frac{x^5}{25}(5 \ln x - 1) + C.$

16. $\int \sqrt{\frac{\arcsin x}{1 - x^2}} \, dx.$

17. $\int x\sqrt{1 - x^2} \arcsin x \, dx = \frac{1}{8}[3x - x^3 - 3(1 - x^2)^{3/2} \arcsin x] + C.$

18. $\int (1 + x) \cos \sqrt{x} \, dx.$

19. $\int \sin 2x \, e^{\sin^2 x} \, dx.$

20. $\int a^x b^{2x} c^{3x} \, dx.$

21. $\int \frac{dx}{(x^2 + 4)^2}.$

22. $\int \frac{x}{\sqrt{2x - x^2}} dx.$

23. $\int \sqrt{a^{1/3} + x^{1/3}} \, dx.$

24. $\int \sqrt{\frac{x}{a - x}} \, dx.$

25. $\int \frac{\sin x}{2 - \sin^2 x} dx.$

26. $\int \sin x \ln (1 + \sin x) \, dx.$

27. $\int \frac{\cos x}{9 - 4 \sin^2 x} dx.$

28. $\int \sin^3 2x \, dx.$

29. $\int \cos^5 3x \, dx.$

30. $\int \cos^5 2x \sin^2 2x \, dx.$

31. $\int \sin^7 3x \cos^2 3x \, dx.$

32. $\int \frac{\sin^5 x}{\cos^2 x} dx.$

33. $\int \frac{\cos^7 2x}{\sin^2 2x} dx.$

34. $\int \tan^4 6x \, dx.$

35. $\int \sec^4 8x \, dx.$

36. $\int \frac{\sec x}{\sec x - \tan x} dx.$

37. $\int \frac{dx}{e^x - a^2 e^{-x}}.$

CHAPTER X

87. Differential of Arc.

In Figure 70 let s represent the length of an arc of the curve

$$y = f(x)$$

from an initial point A of the curve to $P(x,y)$. Then arc $AP = s$ and s is a function of x. For convenience, it is assumed that s increases as x increases. An increment of x produces corresponding increments of y and s and locates a point $P'(x + \Delta x, y + \Delta y)$ on the curve so that the length of the arc $PP' = \Delta s$. Let the length of the chord $PP' = \overline{PP'}$. Then from the figure
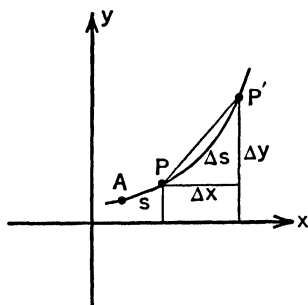
$$PP' = \sqrt{\overline{\Delta x}^2 + \overline{\Delta y}^2}$$

and

$$\frac{PP'}{\Delta x} = \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}.$$

Fig. 70

Multiplying the numerator and the denominator of the first member of the equation by Δs,

$$\frac{\Delta s}{\Delta x}\frac{PP'}{\Delta s} = \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}.$$

Hence,

$$\lim_{\Delta x \to 0}\frac{\Delta s}{\Delta x}\cdot\lim_{\Delta x \to 0}\frac{PP'}{\Delta s} = \lim_{\Delta x \to 0}\sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}$$

and

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2},$$

in which we have assumed that $\lim_{\Delta x \to 0}\dfrac{PP'}{\Delta s} = 1$, as is shown later in this section.

From the definition of a differential of a function,

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

Upon dividing the expression for PP', given above, by Δy instead of by Δx, it can be shown, similarly, that

$$ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy,$$

which will be found useful upon occasions.

The latter two forms for the differential of arc can be written in the equivalent form

$$ds = \sqrt{dx^2 + dy^2},$$

using that $\left(\frac{dy}{dx}\right)^2 = \frac{(dy)^2}{(dx)^2}$ from Chapter III.

In case the equation of the curve is expressed parametrically, so that x and y are expressed as functions of the parameter t, the expression for PP', given above, may be divided by Δt. This yields the following value for the differential of arc:

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

The positive sign used preceding the radical in each of the above differentials is a result of the assumption that s increases with x, with y and with t. Otherwise, the sign might be taken as negative.

Limit of the Ratio of Arc and Chord. It was assumed above that the limit of the ratio of the arc and the chord joining two near points of a curve approached unity as one of those points approached the other as a limit. A formal proof of this statement can be given as follows:

In Figure 71 a tangent is drawn to the given curve at the point P on the curve. At the second near point P' of the curve a perpendicular is erected to the chord PP' intersecting the tangent at the point Q. Let the angle $QPP' = \theta$. From the figure it is obvious that if P and P' are sufficiently near points,
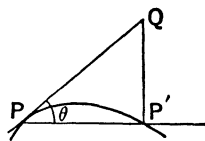
Fig. 71

$$\text{Chord } PP' < \text{Arc } PP' < PQ + P'Q$$

and

$$1 < \frac{\text{Arc } PP'}{\text{Chord } PP'} < \frac{PQ}{PP'} + \frac{P'Q}{PP'}.$$

Also from the figure,

$$\frac{PQ}{PP'} = \sec\theta \quad \text{and} \quad \frac{P'Q}{PP'} = \tan\theta,$$

whence
$$1 < \frac{\text{Arc } PP'}{\text{Chord } PP'} < \sec\theta + \tan\theta.$$

As θ approaches zero, the right-hand member of the inequality approaches unity. Hence, as P' approaches P as a limiting position, the ratio of the arc and the chord approaches unity.

88. Curvature.

A curve which changes its direction rapidly is often spoken of as a sharp curve as compared with one which changes its direction more slowly over an arc of equal length. Such statements express qualitative aspects of a curve. It is our task to find a quantitative expression for the sharpness or flatness of a curve.
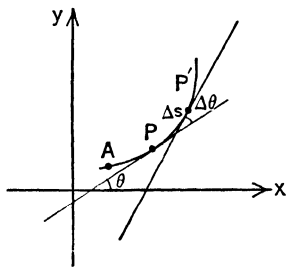
FIG. 72

In Figure 72 let $\Delta\theta$ represent the angle by which the tangent to the curve at the point P has changed its direction as the point of tangency P moves along the curve to a near point P'. The measure of this angle depends, not only on the sharpness of the curve, but also on the arc length $PP' = \Delta s$.

The ratio of the change in the direction of the tangent and the change of arc length is defined as the *average curvature* of the curve over the arc Δs. Hence, letting K denote curvature,

$$\textbf{Average } K = \frac{\Delta\theta}{\Delta s}.$$

The limit of this ratio as the point P' approaches P as a limiting position is defined as the *curvature* of the curve at the point P. Hence,

$$K = \lim_{\Delta s \to 0} \frac{\Delta\theta}{\Delta s} = \frac{d\theta}{ds}.$$

Thus, *the curvature of a curve at any given point is the rate of change of its inclination per unit of its arc length.*

The definition given for curvature is independent of any coordinate system. The change in direction of a tangent may be measured with ref-

erence to any fixed line of the plane. Those properties of a curve which are independent of a coordinate system are known as the *intrinsic proper-ties* of the curve.

To derive the analytic expression for curvature, a coordinate system must be chosen. Assuming such a system, the equation of the curve is expressed in Cartesian coordinates and the direction of the tangent is taken as the slope angle. As a result, the curvature of a curve at any point can be evaluated by means of the coordinates of that point.

In Figure 72, s is measured from the fixed point A on the curve

$$y = f(x)$$

to the point $P(x,y)$. The slope of the tangent at P is

$$\tan \theta = \frac{dy}{dx} \quad \text{and} \quad \theta = \arctan \frac{dy}{dx}.$$

Hence,
$$\frac{d\theta}{dx} = \frac{\dfrac{d^2y}{dx^2}}{1 + \left(\dfrac{dy}{dx}\right)^2}.$$

From Section 87,
$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}.$$

Since,
$$\frac{d\theta}{ds} = \frac{\dfrac{d\theta}{dx}}{\dfrac{ds}{dx}},$$

we have,
$$K = \frac{d\theta}{ds} = \frac{\dfrac{d^2y}{dx^2}}{\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{3/2}}.$$

According to this definition, the curvature of a curve $y = f(x)$ at any point $P_1(x_1,y_1)$ has the same sign as that of $f''(x_1)$. Hence, K is positive or negative according as the curve is concave upward or concave downward at the point P_1. Ordinarily, it is customary to consider curvature as essentially positive. However, as we shall see, there are occasions when we shall desire to retain the sign.

The curvature of a curve $y = f(x)$ vanishes at any point of the curve at which $f''(x) = 0$. Thus, the curvature is zero at an inflection point of the curve.

Radius, Center and Circle of Curvature. The reciprocal of the curvature of a curve at a point is called the *radius of curvature* of the curve at that point. Denoting the radius of curvature by ρ, by definition

$$\rho = \frac{1}{K}.$$

The curvature of a circle is a constant and is equal to the reciprocal of its radius. Thus, if

$$x^2 + y^2 = a^2, \quad K = \frac{1}{a},$$

and the radius of curvature of a circle is its radius. By means of this result, the analytic definition of curvature, when applied to a circle, conforms with the qualitative statement at the beginning of this section: the larger the radius of a circle, the smaller the curvature and the more nearly the circle coincides with its tangent at a point.

FIG. 73

Let a length equal to the radius of curvature of a curve at a point P, for which the second derivative does not vanish, be laid off on the normal to the tangent to the curve at P toward the concave side of the curve. The extremity R of this line segment is known as the *center of curvature* for the point P. The circle whose center is R and whose radius is the reciprocal of the curvature at P, is called the *circle of curvature* at the point P. If the given curve is a circle, the center of curvature is the center of the circle and the circle of curvature is the circle itself. It can be shown by more advanced methods that the circle of curvature of a curve, other than a circle, represents the curve near the point P more nearly than does any other circle.
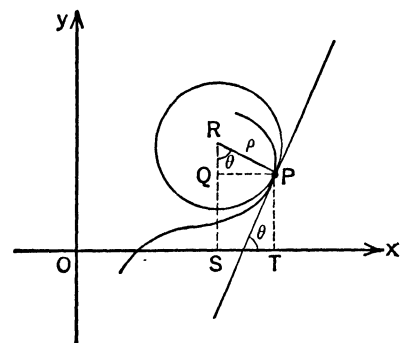
The coordinates of the center of the curvature of a curve $y = f(x)$ at a given point P of the curve are derived by means of Figure 73. Let the coordinates of R be (h,k) and choose the positive value of the radius of curvature ρ, since the curve is concave upward at the point P. Were the curve concave downward at the point, ρ would be taken negative. Then, since $f'(x) = \tan \theta$ and $ds = \sqrt{1 + [f'(x)]^2}\ dx = \sec \theta\ dx$, $\cos \theta = dx/ds$

and $\sin\theta = dy/ds$. Hence,

$$\sin\theta = \frac{f'(x)}{\sqrt{1 + [f'(x)]^2}}, \quad \cos\theta = \frac{1}{\sqrt{1 + [f'(x)]^2}}.$$

From the figure,

$$h = OT - QP = x - \rho\sin\theta,$$
$$k = TP + QR = y + \rho\cos\theta.$$

In these equations we substitute

$$\rho = \frac{\{1 + [f'(x)]^2\}^{3/2}}{f''(x)},$$

assuming that $f''(x) \neq 0$. Substituting the values for $\sin\theta$ and $\cos\theta$, obtained above, we have the coordinates

$$h = x - \frac{f'(x)\,\{1 + [f'(x)]^2\}}{f''(x)},$$

$$k = y + \frac{1 + [f'(x)]^2}{f''(x)}.$$

The curvature, the radius of curvature and the coordinates of the center of curvature of the parabola $y = x^2$ at the point $(\sqrt{2}, 2)$ are found as follows:

$$f'(x) = 2x, \quad f'(\sqrt{2}) = 2\sqrt{2}.$$

$$f''(x) = 2. \quad \text{Hence,} \quad K = \frac{2}{27} \quad \text{and} \quad \rho = \frac{27}{2}.$$

$$h = \sqrt{2} - 9\sqrt{2} = -8\sqrt{2},$$

$$k = 2 + \frac{9}{2} = \frac{13}{2}.$$

Exercise 61

GROUP A

Find the curvature of each of the following curves.

1. $y^2 = 8x$ at $(\frac{9}{8}, 3)$.
2. $3y = x^3$ at $(2, \frac{8}{3})$.
3. $y = x^2 - 3x + 1$ at $(1, -1)$.

4. $y^2 + x = 2$ at $(-2, 2)$.
5. $y = \dfrac{8a^3}{x^2 + 4a^2}$ at $(2a, a)$.

Find the radius of curvature of each of the following curves.

    6. $y^2 = 4 - x$ at $(0,2)$.   **9.** $16x^2 + 9y^2 = 288$ at $(3,4)$.

    7. $y = 3 + x - x^2 - x^3$ at $(0,3)$. **10.** $4y^2 = x^2 + 36$ at $(8,5)$.

    8. $27y^2 = 4x^3$.

Find the coordinates of the center of curvature at any point of each of the following curves.

    11. $x^2 = 4y$.   **12.** $2xy = a^2$.

GROUP B

Find the curvature of each of the following curves.

    13. $y = xe^{-x}$ at $(1, 1/e)$.   **16.** $x^{2/3} + y^{2/3} = a^{2/3}$.

    14. $y = \ln \sec x$.   **17.** $x = t + t^2$, $y = t - t^2$ at $t = 1$.

    15. $\sqrt{x} + \sqrt{y} = \sqrt{a}$.   **18.** $x = \sin \theta$, $y = \cos 2\theta$ at $\theta = \pi/2$.

Find the radius of curvature of each of the following curves.

    19. $y = \arcsin \sqrt{2x - x^2}$ at $x = 1$.

    20. $x = a \cos \theta$, $y = b \sin \theta$ at $\theta = \pi/4$.

    21. $y = x \sin x$ at $x = \pi/2$.

    22. $x = t^2$, $y = 2t^3e^t$ at $t = 1$.

    23. $x = e^t \cos t$, $y = e^t \sin t$ at $t = \pi/2$.

Find the coordinates of the center of curvature at any point of each of the following curves.

    24. $y = \cos x$ at $x = \pi$.

    25. $x = t - t^2$, $y = t + t^2$ at $t = -1$.

    26. $x = \sin 2\theta$, $y = \cos^2 \theta$.

27. Find the coordinates of the point on the curve $y = e^x$ at which the curvature is maximum.

28. Find the coordinates of the point on the curve $x = a(\theta - \sin \theta)$, $y = a(1 - \cos \theta)$ at which the radius of curvature is maximum.

29. Find the coordinates of the point on the curve $y^2 = 4ax$ at which the curvature is maximum.

30. Find the coordinates of the point on the curve $xy = a^2$ at which the curvature is maximum.

GROUP C

Find the radius of curvature for each of the following curves.

    31. $x = a \cos \theta + a\theta \sin \theta$, $y = a \sin \theta - a\theta \cos \theta$.

    32. $y = a \arccos \dfrac{a - x}{a} - \sqrt{2ax - x^2}$ at $x = a/2$.

    33. $y = x \sin \dfrac{1}{x}$ at $x = 2/\pi$.

    34 $y = \dfrac{\sin x}{x}$ at $x = \pi$.

    35. $x = 2 \cos \theta$, $y = \sin^2 \theta$.

    36. $x = a \cos^2 t$, $y = a \sin^3 t$.

    37. $x = 2 \sin^4 \theta$, $y = 2 \cos^4 \theta$ at $\theta = \pi/4$.

38. Find the value of t for which the radius of curvature of $x = 2 \cos^3 t$, $y = 2 \sin^3 t$ is maximum.

39. Find the coordinates of the center of curvature at any point of the curve $y^2 = 2ax$, eliminate x and y from these equations and thus find the equation of the locus of the center of curvature. This curve is called the *evolute* of the curve.

40. A point moves along the arc of the curve $y = x^3$ at the rate of 4 ins. per sec. Approximate the change in direction of motion during the second after it has passed the point $(1,1)$.

89. Length of an Arc of a Curve.

The integral calculus enables us to compute the length of an arc of a plane curve. To do so is frequently called, to *rectify* the curve.

Let s represent the length of the continuous single-valued curve $y = f(x)$ from the point $A(a,a')$ to the point $B(b,b')$ as in Figure 74. If the interval $AB = b - a$ is divided into n equal increments Δx, and if the ordinates of the curve are drawn at each of the division points, the arc AB is divided into n unequal increments Δs. The problem of expressing the length of any increment of arc, Δs_i, presents much the same problem as that of finding the length of the arc AB. However, it is possible to find an approximation ds_i

FIG. 74

for this increment by the use of a theorem known as the Mean Value Theorem which is presented in a later chapter. Since the results of this theorem are not yet available, a different method of approach to the problem is presented.
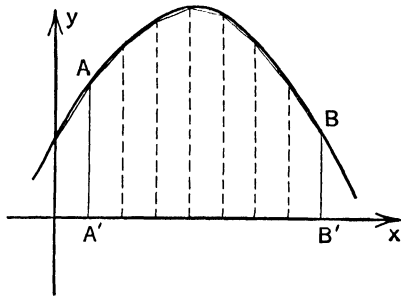
In Section 87 it is shown that the differential of an arc length is

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

Since, in general, $\dfrac{dy}{dx}$ is a function of x,

$$ds = g(x)\, dx.$$

Making use of the indefinite integral,

$$s = \int g(x)\, dx = G(x) + C.$$

Hence, the length of the arc AB is expressed by the definite integral

$$s = \int_a^b g(x) \, dx = G(b) - G(a),$$

or

$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$

Similarly, if y is to be the variable of integration, the second form of the differential of arc length from Section 87 is used, giving

$$s = \int_{a'}^{b'} \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, dy.$$

Again, if the equations of the curve are given in parametric form, where x and y are functions of t and where t_1 corresponds to the point A and t_2, to B,

$$s = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt.$$

To find the circumference of a circle of radius a, we may proceed as follows:

$$s = 4 \int_0^a \sqrt{1 + \frac{x^2}{y^2}} \, dx = 4a \int_0^a \frac{dx}{y}$$

$$s = 4a \int_0^a \frac{dx}{\sqrt{a^2 - x^2}} = 4a \arcsin \frac{x}{a}\bigg]_0^a = 2a\pi.$$

Or, more simply, we may use the parametric equations of the circle,

$$x = a \cos \theta, \quad y = a \sin \theta.$$

Thus,

$$s = 4a \int_0^{\pi/2} \sqrt{\sin^2 \theta + \cos^2 \theta} \, d\theta$$

$$s = 4a \int_0^{\pi/2} d\theta = 2a\pi.$$

Exercise 62

GROUP A

Find the length of the arc of each of the following curves.

1. $y^2 = 4x - x^2$.

2. $y^2 = 8x$ from the vertex to $x = 2$.

3. $y = \ln \sin x$ from $x = \pi/2$ to $x = \frac{3}{4}\pi$.

4. $y = \ln \sec x$ from $x = 0$ to $x = \pi/4$.

5. $x = a \cos^3 \theta$, $y = a \sin^3 \theta$ from $\theta = 0$ to $\theta = \pi/2$.

6. One arch of $x = a(\theta - \sin \theta)$, $y = a(1 - \cos \theta)$.

7. $x^{2/3} + y^{2/3} = a^{2/3}$.

8. $3y^2 = x^2(1 - x)$ over the loop.

9. $y = \ln x$ from $x = 1$ to $x = 2$.

10. $y = e^x$ from $x = 0$ to $x = 1$.

GROUP B

Find the length of the arc of each of the following curves.

11. $y = \ln (1 - x^2)$ from $x = 0$ to $x = \frac{1}{2}$.

12. $x = \cos \theta + \theta \sin \theta$, $y = \sin \theta - \theta \cos \theta$ from $\theta = 0$ to $\theta = \pi$.

13. $8y = x^2 - 8 \ln x$ from $x = 1$ to $x = 2$.

14. $y = 5x^3$ from $x = 0$ to $x = 1$.

15. $y = e^{x/2} + e^{-x/2}$ from $x = 0$ to $x = a$.

16. $y = \ln \dfrac{e^x - 1}{e^x + 1}$ from $x = 1$ to $x = 2$.

17. $x = e^{-2\theta} \sin 2\theta$, $y = e^{-2\theta} \cos 2\theta$ from $\theta = 0$ to $\theta = \pi/2$.

In each of the following problems a particle moves on a curve whose parametric equations are given. Find the distance covered by the particle as specified.

18. $x = 4t - 2t^2$, $y = t^2 - 2t + 2$ during the first two seconds.

19. $x = 1 - 2 \sin 2t$, $y = 2 + 2 \cos 2t$ in traversing the entire curve.

20. $x = 2t - \sin 2t$, $y = \cos 2t$ in traversing one arch of the curve.

90. Area of a Surface of Revolution.

If a plane curve is rotated about a line in its plane, a surface of revolution is generated. The area of such a surface can be computed by integration.

In Figure 75, let the arc AB of the curve $y = f(x)$ be rotated about the x-axis and generate a surface whose area is S. The interval $A'B'$ on the x-axis is divided into n equal parts of length Δx. Planes erected perpendicular to the x-axis at each of these points of division divide the surface into n narrow bands each of which is an increment ΔS of the surface S. Each band is generated by the rotation of an increment Δs of the arc of the curve. Consider one such increment of arc, $\Delta s_i = P_iP_{i+1}$
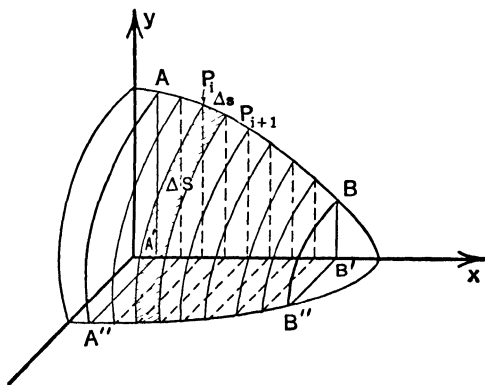
FIG. 75

and the generated increment of area ΔS_i. An approximation to ΔS_i is

$$2\pi\left[\frac{y_i + (y_i + \Delta y_i)}{2}\right] \sqrt{\overline{\Delta x}^2 + \overline{\Delta y_i}^2},$$

which is the surface area of the frustum of the cone generated by the revolution of the chord P_iP_{i+1} about the x-axis. Since the length of the increment of the arc differs from the length of the chord by an infinitesimal of higher order than Δs_i, we write

$$2\pi\left(y_i \, ds_i + \frac{\Delta y_i}{2} \, ds_i\right),$$

as an approximation to ΔS_i. Here again, if we neglect the infinitesimal $\frac{\Delta y_i}{2} \, ds_i$, the *element of surface area* is

$$dS_i = 2\pi y_i \, ds_i,$$

since the sum of all such elements is an arbitrarily close approximation to the surface area. The limit of the sum as n becomes infinite is the surface area. Thus,

$$S = \lim_{n \to \infty} \sum_{i=1}^{i=n} 2\pi y_i \, ds_i$$

and by the fundamental theorem

$$S = 2\pi \int_C y \, ds.$$

The limits of the definite integral, being indicated by C, express that the integral is to be evaluated for limits giving the area of the surface from plane $AA'A''$ to plane $BB'B''$. One of the three forms given for the differential of the arc length is chosen from those given in the previous section, dependent on the choice of the variable of integration.

In the above derivation the curve $y = f(x)$ from A to B was rotated about the x-axis. Consequently, any point P on the curve generates a circle whose center is on the x-axis and whose radius is y. Hence, using x as the variable of integration, the element of surface area is

$$dS = 2\pi y_i \, ds_i = 2\pi y_i\sqrt{1 + [f'(x_i)]^2} \, dx$$

and

$$S = 2\pi \int_{x_1}^{x_2} y\sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx.$$

To find the area of the surface generated by the rotation of the arc $P_1(1,2)$ $P_2(4,4)$ of the parabola $y^2 = 4x$ about the line $y - 2 = 0$, we proceed as follows:
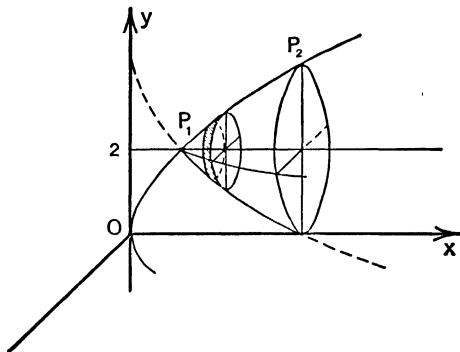
FIG. 76

As shown in Figure 76, a circle of revolution has the radius $(y - 2)$ with its center on the line $y - 2 = 0$. Hence, an element of the surface area is

$$dS = 2\pi(y_i - 2)\, ds_i.$$

and

$$S = \lim_{\Delta y \to 0} \sum_{i=1}^{i=n} 2\pi(y_i - 2)\sqrt{1 + \frac{y_i^2}{4}}\, \Delta y.$$

Since $\dfrac{dx}{dy} = \dfrac{y}{2}$, y is chosen as the variable of integration. Therefore,

$$S = \pi \int_2^4 (y - 2)\sqrt{y^2 + 4}\, dy,$$

$$S = \frac{\pi}{3}\left[(y^2 + 4)^{3/2} - 3y\sqrt{y^2 + 4} - 12\ln(y + \sqrt{y^2 + 4})\right]_2^4$$

and

$$S = \frac{4}{3}\pi\left[4\sqrt{5} - \sqrt{2} + 3\ln\frac{1 + \sqrt{2}}{2 + \sqrt{5}}\right].$$

Exercise 63

GROUP A

Find the area of each of the following surfaces of revolution.

1. A sphere of radius a.

2. A zone cut from a sphere of radius a by two parallel planes a distance of h apart.

3. One arch of $y = \sin x$ rotated about the x-axis.
4. The portion of $y = \ln x$ in the fourth quadrant rotated about the y-axis.
5. The portion of $x^{2/3} + y^{2/3} = a^{2/3}$ above the x-axis rotated about the x-axis.
6. The torus generated by rotating a circle of radius a about a line in its plane b units from the center, $b > a$.
7. The surface generated by rotating a circle about a tangent.
8. The curve $4y = x^3$ from $x = 0$ to $x = 2$ rotated about the x-axis.

GROUP B

Find the area of each of the following surfaces of revolution.

9. The curve $y = e^{x/2} + e^{-x/2}$ from $x = -a$ to $x = a$ about the x-axis.
10. The curve $y = \dfrac{a}{2}(e^{x/a} + e^{-x/a})$ from $x = -k$ to $x = k$ about the tangent at the lowest point.
11. The upper half of the curve $x = 5\cos^3\theta$, $y = 5\sin^3\theta$ rotated about the x-axis.
12. The curve $y^2 = ax$ between $y = 0$ and $y = a$ rotated about the y-axis.
13. One arch of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ rotated about the x-axis.
14. One arch of the cycloid rotated about the tangent at its highest point.
15. The curve $x^2 - y^2 = a^2$ from $x = a$ to $x = 2a$ rotated about the x-axis.
16. The curve $3y^2 = x^3$ from $x = 0$ to $x = 3$ rotated about the x-axis.

91. Mean Value of a Function.

Let $y = f(x)$ be a continuous function from $x = a$ to $x = b$. Divide the interval $b - a$ into n equal parts of length Δx. If the ordinates of the curve are drawn at each of the points of division,

$$y_1 = f(a) \quad \text{and} \quad y_i = f(x_i),$$

where

$$x_i = a + (i - 1)\,\Delta x.$$

The sum of all such ordinates divided by the number of them,

$$\frac{\sum\limits_{i=1}^{i=n} y_i}{n} = \frac{\sum\limits_{i=1}^{i=n} f(x_i)}{n},$$

is the arithmetic mean of the ordinates drawn. If the numerator and denominator of this fraction is multiplied by Δx, and if the limit is taken as n becomes infinite, the arithmetic mean M of the ordinates is

$$M = \frac{\lim\limits_{n\to\infty} \sum\limits_{i=1}^{i=n} f(x_i)\,\Delta x}{\lim\limits_{n\to\infty} n\,\Delta x} = \frac{\int_a^b f(x)\,dx}{b - a},$$

since $n\,\Delta x$ is equal to the constant $b - a$.

The expression written for M is called the *mean value* of the function with respect to its variable over the given range.

To find the mean length of the ordinates of points on the first quadrant arc of the circle $x^2 + y^2 = a^2$, assuming the points equally spaced along the circle, we proceed as follows:

The mean of n ordinates is

$$\frac{\sum\limits_{i=1}^{i=n} y_i}{n} = \frac{\sum\limits_{i=1}^{i=n} y_i \, \Delta s}{n \, \Delta s},$$

where Δs is supplied in numerator and denominator, since the ordinates are equally spaced along the arc of the circle.   Also, since,

$$n \, \Delta s = \frac{a}{2}\pi, \quad \text{and} \quad ds = \frac{a}{y}\,dx,$$

$$M = \frac{2}{a\pi}\int_C y\,ds = \frac{2}{\pi}\int_0^a dx = \frac{2a}{\pi}.$$

Exercise 64

GROUP A

1. Find the mean width of a semicircle, assuming that the widths are taken as verticals to the diameter and equally spaced along it.
2. Find the mean width of the segment of $y^2 = 4x$ to $x = 3$, assuming that the widths are taken perpendicular to the x-axis and equally spaced along it.
3. Find the mean width of the segment of the parabola $y^2 = 4x$ to $x = 4$, assuming that the widths are taken perpendicular to the y-axis and equally spaced along it.
4. Find the mean width of an arch of the curve $y = \sin x$, assuming that the widths are taken parallel to the x-axis and equally spaced along the vertical.
5. Line segments are drawn from the point $(2,0)$ to points which are equally spaced on the circle $x^2 + y^2 = 16$.   Find the mean of the squares of the lengths of the line segments.
6. Find the mean width of the area between $y = x^2$ and $x = y^2$, assuming that the widths are taken vertically and equally spaced along the x-axis.
7. Find the mean width of the loop of the curve $y^2 = x^2(3 - x)$, assuming that the widths are taken vertically and equally spaced along the x-axis.
8. If rectangles are inscribed in the circle $x^2 + y^2 = a^2$, prove that the mean area when the altitudes are equally spaced along the x-axis, is equal to the mean area when the bases are equally spaced along the y-axis.

GROUP B

9. If rectangles are inscribed in the ellipse $b^2x^2 + a^2y^2 = a^2b^2$, prove that the mean area is the same whether the altitudes are equally spaced along the x-axis or the bases are equally spaced along the y-axis.

10. Find the mean area of isosceles triangles inscribed in a circle of radius a, assuming that they have a common vertex and that the bases are equally spaced along the diameter through the vertex.

11. Find the mean area of the isosceles triangles in Problem 10, assuming that the bases are equally spaced along the arc of the circle.

12. Find the mean area of the sections of a right circular cone of altitude h and base a, assuming that the sections are perpendicular to the axis and equally spaced along it.

13. Find the mean volume of the cylinders inscribed in a sphere of radius a, assuming that the bases of the cylinders are equally spaced along a diameter of the sphere.

14. Find the mean velocity of a body falling freely from rest during the time a secs, assuming that the velocity is averaged with respect to time.

15. Find the mean velocity of a body falling freely from rest through the distance b ft., assuming that the velocity is averaged with respect to distance.

16. A particle has simple harmonic motion given by $s = a \sin bt$. Find the mean velocity of the particle during the time of motion from the mean position to an extreme.

17. If $y = x \ln x$, find the mean value of y from $x = 1$ to $x = 4$, assuming equal differences of x.

18. If the volume v of a gas varies inversely as the pressure p, find the average pressure as the gas expands from a cu ft. to b cu. ft.

19. If the angular velocity of a rotating wheel is proportional to the cube of the time, find the average velocity from t_1 to t_2.

20. Find the length of the curve $y = \ln \cos x$ from $x = 0$ to $x = \pi/3$.

92. Double Integrals.

The integral of an integral, or a *definite iterated integral* is represented by the symbol

$$\int_a^b \left[\int_{y_1}^{y_2} f(x,y) \, dy \right] dx.$$

We shall call this integral a *double integral.*

In the double integral, as written, the *inner integral belongs with the inner differential dy* and the *inner limits* are either constants or functions of x alone. The *outer integral belongs with the outer differential dx* and the *outer limits* are constants. During the first integration with respect to y, x *is held fixed.* The result is a function of x alone. If we let $F(x)$ represent the evaluation of the inner integral, it is possible to evaluate the double integral as follows:

$$\int_a^b \int_{y_1}^{y_2} f(x,y) \, dy \, dx = \int_a^b F(x) \, dx = E(b) - E(a).$$

We shall have occasion, also, to evaluate a double integral by an integration with respect to x first, followed by an integration with respect

to y. In this case, the inner limits are constants or functions of y and y is held fixed during the first integration. The result of this integration is a function of y which may be represented by $G(y)$. The outer limits are constants. The evaluation of a double integral with this order of integration, may be represented as follows:

$$\int_c^d \int_{x_1}^{x_2} f(x,y) \, dx \, dy = \int_c^d G(y) \, dy = H(d) - H(c).$$

The evaluation of the two following double integrals will illustrate the technique of double integration.

$$\int_1^2 \int_{y^2}^{2y} 3 \, dx \, dy = \int_1^2 3x \Big]_{y^2}^{2y} dy = \int_1^2 3(2y - y^2) \, dy = 2.$$

$$\int_0^2 \int_0^{x^3} xy \, dy \, dx = \frac{1}{2} \int_0^2 xy^2 \Big]_0^{x^3} dx = \frac{1}{2} \int_0^2 x^7 \, dx = 16.$$

93. Plane Area by Double Integration.

The symbol for a *double sum* is

$$\sum_{j=1}^{j=n} \sum_{i=1}^{i=m} x_i y_j.$$

The double summation sign indicates that *both variables x and y*, enter into the quantity which is to be summed. The indicated sum, written out completely for $m = 4$ and $n = 3$ is

$$\sum_{j=1}^{j=3} (x_1 + x_2 + x_3 + x_4)y_j = (x_1 + x_2 + x_3 + x_4)y_1$$
$$+ (x_1 + x_2 + x_3 + x_4)y_2 + (x_1 + x_2 + x_3 + x_4)y_3.$$

The area between two curves

$$y = f_1(x) \quad \text{and} \quad y = f_2(x), \quad \text{or} \quad x = g_1(y) \quad \text{and} \quad x = g_2(y),$$

as drawn in Figure 77, may be expressed as the limit of a double sum.

As shown in the figure, it is assumed that the curves intersect at the points $A(a',a'')$ and $B(b',b'')$ and at no other points. Thus the horizontal line segment between $P_j'(x_j',y_j)$ on curve (1) and $P_j''(x_j'',y_j)$ on curve (2) has the length $(x_j'' - x_j')$ for all positions between A and B.

First, the area between the two curves is divided into n horizontal strips, each of width Δy. Any element of area ΔS_j differs from the area of the corresponding strip whose base is $P_j'P_j''$ by infinitesimals of higher order

than $(x_i'' - x_i') \Delta y_j$. Then this jth strip is divided into m rectangles by vertical lines from P_j' to P_j'' at equal intervals of Δx. The area of the ith rectangle in the jth strip is an *element of area*,

$$dS = \Delta x_i \, \Delta y_j.$$
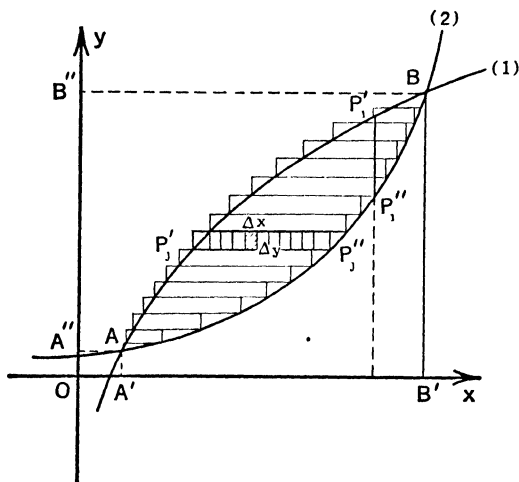
FIG. 77

Holding y_j and Δy_j fixed, the position and width of the jth strip remains constant. The area of this strip is the sum of the m rectangles, or

$$\sum_{i=1}^{i=m} \Delta x_i \, \Delta y_j.$$

The sum of the areas of the n strips is an approximation to the area between the two curves from $y = a''$ to $y = b''$ for any values of m and n. This approximation is the double sum

$$\sum_{j=1}^{j=n} \sum_{i=1}^{i=m} \Delta x_i \, \Delta y_j.$$

The area between the curves is the limit of this double sum as m and n become infinite.

An extension of the Fundamental Theorem given in Section 50, Chapter VI, permits us to express the limit of a double sum by a double integral with appropriate limits. Thus, the area between the two curves is

$$S = \lim_{m,n\to\infty} \sum_{j=1}^{j=n} \sum_{i=1}^{i=m} \Delta x_i \, \Delta y_j = \int_{a''}^{b''} \int_{g_1(y)}^{g_2(y)} dx \, dy.$$

Similarly, the area may be expressed by the double integral

$$S = \int_{a'}^{b'} \int_{f_1(x)}^{f_2(x)} dy \, dx.$$

In writing the latter double integral for the area between the two curves, it should be noted that any vertical segment $P_i{}''(x_i,y_i'') \, P_i{}'(x_i,y_i')$, must not intersect either of the curves between those two points.

The choice between the two possible orders of integration may often be motivated by the desire to evaluate the simpler integrals.   More often, however, the relative position of the two curves may require that a particular order be chosen.   A case in point is illustrated in finding the area between the two curves

$$(1) \quad 9y = x^3 \quad \text{and} \quad (2) \quad y = x^2 - 2x.$$
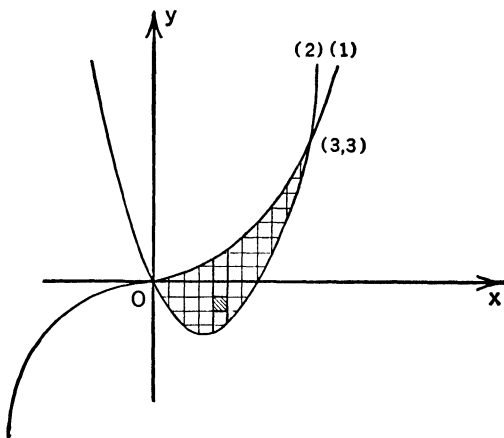
FIG. 78

The two curves are drawn in Figure 78 from which it is clear that the integration with respect to y must be performed first.   The solution is carried out as follows:

$$S = \int_0^3 \int_{y_1}^{y_2} dy \, dx = \int_0^3 y \Big]_{x^2-2x}^{x^3/9} dx$$

$$S = \frac{1}{9} \int_0^3 (x^3 - 9x^2 + 18x) \, dx = \frac{9}{4}.$$

Exercise 65

GROUP A

Find each of the following areas by double integration.

1. Bounded by $y = 2x$ and $y^2 = x^3$.
2. Inside $x^2 - 8y + y^2 = 0$ and outside $x^2 = 4y$.
3. Under one arch of the cycloid $x = a(\theta - \sin \theta)$, $y = a(1 - \cos \theta)$.
4. Enclosed by the curve $x = a \cos^3 \theta$, $y = a \sin^3 \theta$.
5. Bounded by $y = \sin x$, $y = \cos x$ and the y-axis.
6. The loop of the curve $y^2 = x^2(4 - x)$.
7. The area between $y^2 = 2x$ and $x^2 = 16y$.
8. The loop of $a^2x^2 = y^2(a + y)$.

GROUP B

Find each of the following areas by double integration.

9. Inside the curves $x^2 - 4ax + y^2 = 0$ and $y^2 = 2ax$.
10. Bounded by $x^2 = 4ay$ and $y = \dfrac{8a^3}{x^2 + 4a^2}$.
11. Bounded by $y = xe^{-x}$, the x-axis and the ordinates of the maximum point and the inflection point of the curve.
12. Bounded by $y = \sin ax$, $y = \cos ax$ and the x-axis for the first arches of the curves.
13. Bounded by $y^2(x^2 + a^2) = a^2x^2$ and its asymptote in the first quadrant to $x = 2a$.
14. Enclosed by the curve $x = a \cos \theta$, $y = b \sin^3 \theta$.
15. The two areas in the first quadrant between the curves $y^2 = x^3$ and $x^2 + y^2 = 12$.
16. Bounded by $27y^2 = 16x^3$ and $y^2 = 8(5 - x)$.
17. Bounded by $a^2y^2 - x^2y^2 = a^2b^2$ and its asymptotes.
18. Find the length of the curve $y = \frac{2}{3}(1 + x^2)^{3/2}$ from $x = 0$ to $x = 3$.
19. Find the area of the surface generated by rotating the part of the curve $y = e^{-x}$ lying in the first quadrant about the x-axis.
20. If the angular velocity of a rotating wheel is proportional to the square of the time, if it starts from rest and if in 2 minutes it has an angular velocity of 200 revolutions per minute, find the average angular velocity during the 2 minutes of rotation.

94. Volume of Revolution by Double Integration.

Consider the area bounded by the curves

$$y = f_1(x) \quad \text{and} \quad y = f_2(x),$$

as shown in Figure 79. We wish to find the volume which is generated by the rotation of this area about the x-axis.

The area between the two curves is divided into rectangular elements, each Δx by Δy, by drawing equally spaced lines parallel to the x-axis and

parallel to the y-axis. If the ith element in the jth strip is rotated about the x-axis, an element of volume is obtained. As in previous instances, an approximation to the volume of this element is

$$dV = 2\pi y_j \, \Delta x_i \, \Delta y_j.$$

The summation,

$$\sum_{i=1}^{i=m} (2\pi \, \Delta x_i) y_j \, \Delta y_j,$$

holding the jth horizontal strip fixed, is the volume of a cylindrical element.
The summation,

$$\sum_{j=1}^{j=n} (2\pi y_j \, \Delta y_j) \, \Delta x_i,$$

holding the ith vertical strip fixed, is the volume of a ring element.
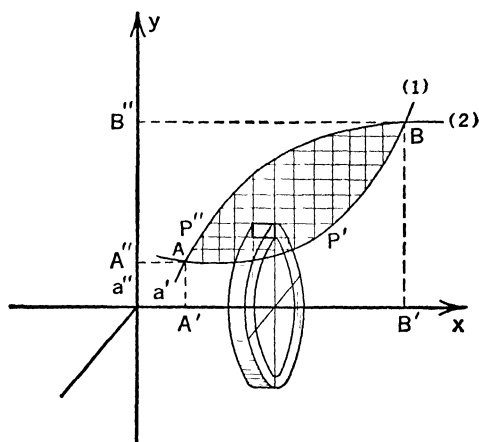
FIG. 79

The double sum of the elements of volume dV is an arbitrarily close approximation to the required volume. The volume is the limit of this sum. By the fundamental theorem,

$$V = \lim_{\substack{n \to \infty \\ m \to \infty}} \sum_{j=1}^{j=n} \left[\sum_{i=1}^{i=m} 2\pi \, \Delta x_i \right] y_j \, \Delta y_j = 2\pi \int_{a''}^{b''} \int_{g_1(y)}^{g_2(y)} y \, dx \, dy,$$

$$V = \lim_{\substack{m \to \infty \\ n \to \infty}} \sum_{i=1}^{i=m} \left[\sum_{j=1}^{j=n} 2\pi y_j \, \Delta y_j \right] \Delta x_i = 2\pi \int_{a'}^{b'} \int_{f_1(x)}^{f_2(x)} y \, dy \, dx.$$

We shall find the volume generated by rotating the area in the first

quadrant bounded by $y = x^2 + 1$, the y-axis and $y - 2 = 0$ about the line $y + 1 = 0$.

The radius of revolution of any element of area, Δx by Δy, is $(y_i + 1)$ as shown in Figure 80. Hence, the element of volume is

$$dV = 2\pi(y_i + 1)\, \Delta y_i\, \Delta x_i.$$
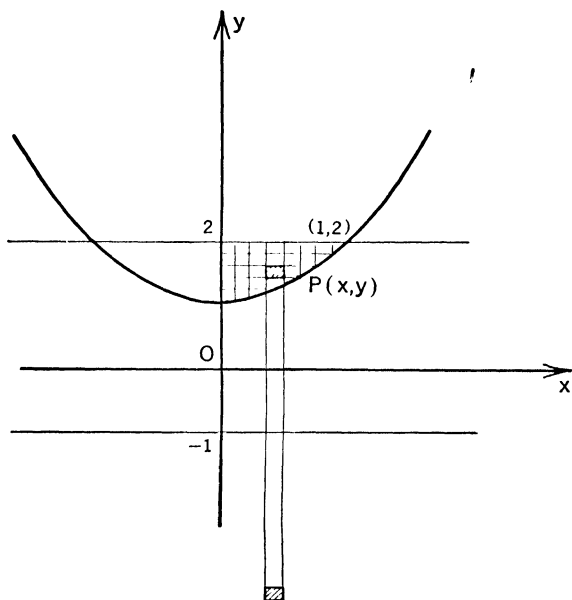
FIG. 80

If y_i and Δy_i are held fixed first, we find the volume as follows:

$$V = 2\pi \int_1^2 \int_0^{\sqrt{y-1}} (y + 1)\, dx\, dy.$$

$$V = 2\pi \int_1^2 (y + 1)x \Big]_0^{\sqrt{y-1}} dy = 2\pi \int_1^2 (y + 1)\sqrt{y - 1}\, dy.$$

But if x_i and Δx_i are kept fixed first, it is found that the integral can be evaluated more readily. This is done as follows:

$$V = 2\pi \int_0^1 \int_{x^2+1}^2 (y + 1)\, dy\, dx = \pi \int_0^1 (y^2 + 2y) \Big]_{x^2+1}^2 dx$$

$$V = \pi \int_0^1 (5 - 4x^2 - x^4)\, dx = \frac{52}{15}\pi.$$

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Exercise 66

GROUP A

Find the volume of each of the following solids of revolution by double integration.

1. A circle of radius a rotated about a diameter.
2. The area bounded by $y^2 = 4x$ and $x = 4$ rotated about the x-axis.
3. The area between $y^2 = 4x$ and $x = 4$ rotated about the y-axis.
4. The area bounded by $y^2 = x^3$ and $x = 4$ rotated about the x-axis.
5. The area in the first quadrant between $y^2 = 2x^3$ and its tangent at $(2,4)$ rotated about the y-axis.
6. The area bounded by $y^2 = 4ax$, $y^2 = 4a^2 - 4ax$ and the x-axis rotated about the y-axis.
7. The area bounded by $y^2 = 4(2 - x)$ and the y-axis rotated about the line $x = 2$.
8. The area enclosed by $\sqrt{x} + \sqrt{y} = \sqrt{a}$ and the coordinate axes rotated about the line $x + a = 0$.

GROUP B

Find the volume of each of the following solids of revolution by double integration.

9. The oval of the curve $y^2 = x(x - 1)(x - 2)$ about the x-axis.
10. The area bounded by $y^2 = x(x - 2)(x - 3)$, the x-axis and $x = 4$ rotated about the x-axis.
11. The area in the first quadrant bounded by $y = (x + 1)^2$, $x = 1$ and $y = 1$ rotated about the y-axis.
12. The area of a circle of radius a rotated about a line in its plane b units from the center, where $b > a$.
13. The area of a circle of radius a rotated about a tangent.
14. The area under one arch of $y = \sin 2x$ rotated about $y + 1 = 0$.
15. The area bounded by $y(x^2 + 4) = 8$, the y-axis and $y = 1$ rotated about the y-axis.
16. The area of the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ rotated about the x-axis.
17. The area of an ellipse rotated about the y-axis.
18. The area bounded by $b^2x^2 - a^2y^2 = a^2b^2$, $y = b$ and the x-axis rotated about the y-axis.
19. The area bounded by $y = \ln x$, $x = 2$ and the x-axis rotated about the y-axis.
20. The area bounded by $y = 2e^{2x}$, $y = e^x$, $x = 1$ and the y-axis rotated about $y = 1$.

GROUP C

Solve each of the following problems by double integration.

21. Find the force exerted on a circular water gate of radius a vertically submerged with its upper edge tangent to the surface of the water.
22. Find the force exerted on a vertical parabolic water gate, base b and altitude a, submerged so that the surface of the water is tangent to vertex of the parabolic segment.

23. Find the work necessary to pump a hemispherical bowl full of water 10 feet above the surface if the radius of the surface is 5 feet.

24. A parabolic bowl is formed by rotating a parabolic segment, base b and altitude a, about its axis. If such a bowl stands so that its vertex is the lowest point and is at a distance of c from the ground, find the work necessary to fill the bowl with water from the ground by a pipe in the bottom.

25. Find the mean value of the square of the distance of all points within a square of side $2a$ from the center of the square.

CHAPTER XI

95. Law of the Mean.

Let the curve $y = f(x)$ be drawn in the Figure 81, where $f(x)$ is a differentiable, continuous and single-valued function in the interval $a \leqq x \leqq b$. From the figure it is obvious that at some point $P_1(x_1, y_1)$ of the curve between $Q[a, f(a)]$ and $R[b, f(b)]$, the tangent to the curve is parallel to the secant QR.

The slope of the secant QR is

$$\frac{BR - AQ}{AB} = \frac{f(b) - f(a)}{b - a}.$$

The slope of the tangent at P_1 is $f'(x_1)$. Hence, it follows that

$$\frac{f(b) - f(a)}{b - a} = f'(x_1)$$

or, $f(b) - f(a) = (b - a)f'(x_1).$
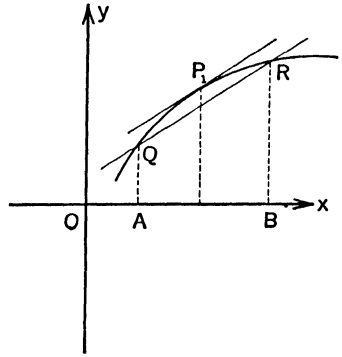
FIG. 81

This is known as the *law of the mean* or, the *Mean Value Theorem* which may be stated as follows:

If a function $f(x)$ is continuous and single-valued in the interval $a \leqq x \leqq b$, and if its first derivative exists in this interval, then there exists at least one value x_1 of x such that

$$f(b) = f(a) + (b - a)\, f'(x_1),$$

where $a < x_1 < b$.

Another useful form of the law of the mean is sometimes written by letting

$$a = x, \quad b = x + \Delta x$$

and by denoting a positive number less than unity $^\lambda$, θ. Thus,

$$f(x + \Delta x) = f(x) + \Delta x\, f'(x + \theta\, \Delta x),$$

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$0 < \theta < 1$. In this statement of the theorem, we have assumed that $f(x)$ is continuous within the interval Δx. It is also assumed that

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

exists. It may be observed that the existence of a derivative $f'(x)$ for every value of x within an interval, guarantees the continuity of $f(x)$ within that interval.
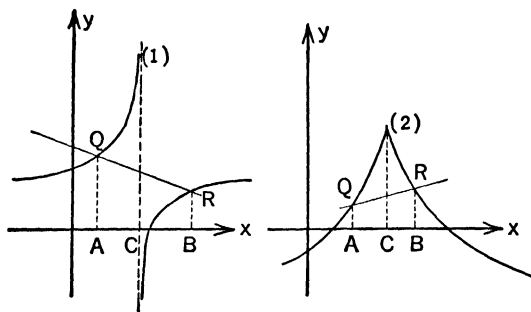
FIG 82

The curve (1) drawn in Figure 82 has a discontinuity at $x = c$. The derivative does not exist for $x = c$. Consequently, there is no point between $x = a$ and $x = b$ at which the tangent is parallel to the secant QR. Again, the curve (2) in Figure 82 has a cusp at $x = c$, $f'(c)$ does not exist and there is no point on the curve between Q and R at which the tangent is parallel to QR.

96. Rolle's Theorem.

A special case of the mean value theorem may be stated as follows: If a function $f(x)$ is continuous and single valued in the interval $a \leqq x \leqq b$, if its first derivative exists and if $f(a) = f(b) = 0$, there exists at least one value of x, as $x = x_1$, for which $f'(x_1) = 0$, where $a < x_1 < b$. This theorem is known as *Rolle's theorem*.

Geometrically, this theorem states that the curve $y = f(x)$, under the conditions assumed, has at least one tangent parallel to the x-axis between the points $(a,0)$ and $(b,0)$. Consequently, such a curve has at least one critical point between the two given points.

Exercise 67

1. Find the coordinates of the point on the curve $y^2 = 4x$ at which the tangent is parallel to the chord $P_1(1,2), P_2(4,4)$.

2. Find the equations of the secant through the points $(1,0)$ and $(2,6)$ of the curve $y = x^3 - x$ and the tangent parallel to it.

3. If the abscissas of points A and B on the curve $y = \ln x$ are 1 and 3, respectively, find the coordinates of the point on the curve at which the tangent is parallel to AB.

4. If $f(x) = x^2$, verify the statement $f(4) = f(0) + 4f'(x_1)$ and find the value of x_1.

5. If $f(r) = \dfrac{1}{x - 1}$, investigate the existence of x_1 such that $f(2) = f(0) + 2f'(x_1)$.

6. Show that $f(0) = f(4) = 0$ for the function $f(x) = 4x - x^2$. Illustrate the validity of Rolle's theorem with this function.

7. Given the function $f(r) = \dfrac{1 - x^2}{x^2}$. Find the values of x for which $f(x) = 0$. Show that Rolle's theorem does or does not apply for this function in the interval between the roots of $f(x) = 0$.

Investigate the validity of the mean value theorem for each of the following functions in the specified intervals.

    8. $y = e^x$ from $x = 0$ to $x = 2$.
    9. $y = \ln (x - 1)$ from $x = 1$ to $x = 3$.
   10. $xy = 1$ from $x = -1$ to $x = 1$.
   11. $y = e^{-x}$ from $x = -1$ to $x = 1$.
   12. $y = \cos x$ from $x = 0$ to $x = \pi/2$.
   13. $y = \tan x$ from $x = 0$ to $x = \pi$.
   14. $y = x^2 - 1$ from $x = -1$ to $x = 1$.

97. " Indeterminate Forms."

The expression $\dfrac{x^2 - 4}{x - 2}$ is defined for all values of x save for $x = 2$ for which it has no meaning. This is because the denominator is then zero. That the numerator is also zero for that value of x, neither helps nor hinders the fact that the expression has no meaning for $x = 2$. By the application of the general rules of operation with limits, it is impossible to evaluate the expression as x approaches 2, since both numerator and denominator approach zero. However, since

$$\frac{x^2 - 4}{x - 2} = x + 2,$$

it becomes evident that the limit of the quotient is

$$\lim_{x \to 2} (x + 2) = 4.$$

Such functions as the one given above for $x = 2$, are generally said to have an "*indeterminate form*." Historically, this designation arose through a vagueness of understanding of the problem involved. Actually, a function of the type given may be undefined for certain values of the variable. Nevertheless, its limit may exist as the variable approaches those values. It is sometimes important to find the limits of various expressions which assume the form illustrated, for certain values of the variable.

Let $f(x)$ and $g(x)$ be two functions each of which vanishes for $x = a$, or each of which becomes infinite for $x = a$. Thus,

$$f(a) = 0 \quad \text{and} \quad g(a) = 0,$$

or,
$$f(a) = \infty \quad \text{and} \quad g(a) = \infty.$$

Under such circumstances the quotient of the two functions has no meaning. We wish to investigate the limit of the quotient of two such functions as x approaches a for the first case only. The method of evaluation for the second case is the same. However, the proof for the latter is omitted.

The first and the second quotients assume the so-called "indeterminate forms"

$$\frac{f(a)}{g(a)} = \frac{0}{0} \quad \text{and} \quad \frac{f(a)}{g(a)} = \frac{\infty}{\infty},$$

respectively.

Suppose that $f(x)$ and $g(x)$ have continuous derivatives at $x = a$ and that $f(a) = g(a) = 0$, but that $g'(a) \neq 0$. Then, from Section 95, where $b = x$, we have

$$f(x) = f(a) + (x - a)f'(x_1)$$

and
$$g(x) = g(a) + (x - a)g'(x_2),$$

where $a < x_1 < x$ and $a < x_2 < x$. Since $g'(a) \neq 0$ and $g'(x)$ is continuous at $x = a$, then $g'(x_2) \neq 0$ if x is sufficiently near a. Therefore, for x near a, but different from a, $g(x)$ is different from zero and may be used as the denominator in the quotient. Hence,

$$\frac{f(x)}{g(x)} = \frac{(x - a)f'(x_1)}{(x - a)g'(x_2)} = \frac{f'(x_1)}{g'(x_2)}.$$

As x approaches a, $f'(x_1)$ approaches $f'(a)$ and $\cdot g'(x_2)$ approaches $g'(a)$. Therefore

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} = \frac{f'(a)}{g'(a)}.$$

If the quotient of the two first derivatives also assumes one of the indeterminate forms, the same procedure is followed, obtaining the second derivatives of the original functions. In fact, the process may be repeated as many times as necessary to obtain the quotient of two derivatives which is not indeterminate. In each case of the evaluation of a limit of a quotient, the numerator and denominator are differentiated *separately* and the limit of the quotient of those derivatives is taken.

The following fractions, as x approaches zero, assume the form $\dfrac{0}{0}$:

$$\frac{\sin x}{x}, \qquad \frac{\tan x}{x}.$$

Differentiating numerators and denominators, and taking the limits,

$$\lim_{x \to 0} \frac{\cos x}{1} = 1, \qquad \lim_{x \to 0} \frac{\sec^2 x}{1} = 1.$$

The limit of the first function was found geometrically in Section 58, which was necessary in order to differentiate the sine. Consequently, this evaluation does not constitute a real verification of the limit.

There are other indeterminate forms of which we shall consider but one,

$$0 \cdot \infty.$$

Assume the product of two functions $f(x)$ and $g(x)$ and suppose that as x approaches a, one function approaches zero while the other function increases without limit. The product of the two functions takes the form $0 \cdot \infty$. However, if we write

$$f(x) \cdot g(x) = \frac{f(x)}{\dfrac{1}{g(x)}},$$

the last quotient takes the form $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$. Consequently, the limit of the quotient of the derivatives of numerator and denominator may be taken as before.

The expression

$$x \ln x,$$

as x approaches zero, takes the form $0 \cdot \infty$. Following the usual procedure, we have

$$\lim_{x \to +0} \frac{\ln x}{\dfrac{1}{x}} = \lim_{x \to +0} \frac{\dfrac{1}{x}}{-\dfrac{1}{x^2}} = \lim_{x \to +0} (-x) = 0.$$

Exercise 68

Evaluate each of the following or show that the limits do not exist.

1. $\lim\limits_{x \to 0} \dfrac{x - x^2}{e^x - 1}$.

2. $\lim\limits_{x \to 9} \dfrac{\sqrt{x} - 3}{x - 9}$.

3. $\lim\limits_{x \to \infty} \dfrac{x - 3}{x^2 + 2}$.

4. $\lim\limits_{x \to 2} \dfrac{x - 2}{\ln (x - 1)}$.

5. $\lim\limits_{x \to 0} \dfrac{x}{1 - e^x}$.

6. $\lim\limits_{x \to \infty} \dfrac{\ln x}{x^2}$.

7. $\lim\limits_{x \to \pi/2} \dfrac{1 - \sin x}{\cos x}$.

8. $\lim\limits_{x \to 0} \dfrac{e^x - 2^x}{x^2}$.

9. $\lim\limits_{x \to 0} (1 + x)^{1/x}$,

by taking the $\lim\limits_{x \to 0} (\ln y)$.

10. $\lim\limits_{x \to 0} \dfrac{\tan 2x}{\ln (1 + x)}$.

11. $\lim\limits_{x \to 0} \dfrac{\sin 2x - \sin x}{\tan 3x}$.

12. $\lim\limits_{x \to 0} \dfrac{1 - e^x}{xe^x + e^x - 1}$.

13. $\lim\limits_{x \to 0} x^2 \ln x^4$.

14. $\lim\limits_{x \to \infty} \dfrac{\ln^2 x}{x^2}$.

15. $\lim\limits_{x \to 2} \dfrac{x - 2}{\ln x - \ln 2}$.

16. $\lim\limits_{x \to \infty} \dfrac{x^n}{e^x}, \; n > 0$.

17. $\lim\limits_{x \to 0} (1 + 2x)^{1/x}$,

by taking the $\lim\limits_{x \to 0} (\ln y)$.

18. $\lim\limits_{x \to 1} \left[\dfrac{1}{\ln x} - \dfrac{x}{x - 1} \right]$.

CHAPTER XII

98. Infinite Series.

This chapter is devoted to an elementary study of infinite series because it is desirable for the student to have some acquaintance with these forms and their applications. This study is limited in its scope for the reason that it is considered inadvisable at this stage to demand the rigor and the completeness to be found in a more advanced treatise. Those who pursue the study of mathematics beyond this course will desire a more exhaustive treatment of the subject.

A *finite series* of n terms is defined by the expression

$$u_1 + u_2 + u_3 + \cdots + u_n = \sum_{i=1}^{i=n} u_i,$$

where n is a positive integer. Ordinarily, each series is formed by some definite law which may be given by means of the nth term.

The following arithmetic and geometric series are examples of finite series where the nth term is written in each series.

$$S_1 = 3 + 5 + 7 + \cdots + [3 + 2(n - 1)].$$
$$S_2 = 3 + 6 + 12 + \cdots + [3(2)^{n-1}].$$

If the number of terms in a series is unlimited, the series is known as an *infinite series*. Such a series is defined by the expression

$$u_1 + u_2 + u_3 + \cdots + u_n + \cdots.$$

The *general term* of a series is usually taken as the nth term. If the general term of a series is known, the law of formation of its terms is known. If, however, the first few terms of the series are known and if it is assumed that the law of formation for succeeding terms is the same as for the first ones, the general term may be formulated. The formulation of the general term for many series presents difficulties. The first terms and the general terms are written for the following infinite series.

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} + \cdots.$$

$$\frac{1}{1 \cdot 2} + \frac{1}{3 \cdot 4} + \frac{1}{5 \cdot 6} + \cdots + \frac{1}{2n(2n - 1)} + \cdots.$$

99. Sum of an Infinite Series.

The *sum S of an infinite series* is defined as the limit of the sum S_n of the first n terms as n increases without limit, provided that such a limit exists. Thus,

$$S = \lim_{n \to \infty} S_n = \lim_{n \to \infty} \sum_{i=1}^{i=n} u_i.$$

The sum of the first n terms of the geometric series is

$$S_n = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} = \frac{a - ar^n}{1 - r}.$$

Hence,

$$S = a + ar + ar^2 + \cdots + ar^{n-1} + \cdots = \lim_{n \to \infty} \frac{a - ar^n}{1 - r}$$

is the sum of the infinite series, provided that it exists.

If $r < 1$, $\displaystyle \lim_{n \to \infty} ar^n = 0.$

In this case the sum exists and its value is given by

$$S = \frac{a}{1 - r}.$$

If $r > 1$, $\displaystyle \lim_{n \to \infty} ar^n = \infty.$

In this case the sum does not exist.   If $r = 1$, the series is $a + a + a + \cdots$, and the sum of the first n terms is $S = an$.   If $a = 0$, $S = 0$.   But if $a \neq 0$, the sum of the infinite series is not defined.

100. Convergence and Divergence.

An infinite series is said to have a sum only in case the limit of the sum of the first n terms exists, as the number of terms is increased without limit. An infinite series which has a sum is called a *convergent series*.   An infinite series which does not possess a sum is called a *divergent series*.

The sum of the first n terms of the infinite geometric series

$$1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots + \frac{1}{3^{n-1}} + \cdots$$

is

$$S_n = \frac{3}{2}\left(1 - \frac{1}{3^n}\right).$$

And since, $$S = \lim_{n \to \infty} S_n = \tfrac{3}{2},$$

the series is convergent. However, the sum of the first n terms of the infinite arithmetic series

$$1 + 3 + 5 + 7 + \cdots + (2n - 1) + \cdots$$

is $$S_n = \frac{n}{2}[2a + d(n - 1)] = n^2.$$

And since, $$\lim_{n \to \infty} S_n = \infty,$$

the series is divergent.

Divergent series are of no importance in elementary applications. However, convergent series are useful in various computational problems. Consequently, in working with series it is essential that the convergence or divergence be established. As we progress in the study of series, it is found that is is often unnecessary, indeed often impossible, to find the sum of an infinite series.

A *necessary condition* for the convergence of an infinite series is that the *limit of the general term be zero,*

$$\lim_{n \to \infty} u_n = 0.$$

This condition is satisfied in every convergent series. However, this necessary condition *is not also a sufficient condition* for the convergence of a series. If the condition is not satisfied, that is,

$$\lim_{n \to \infty} u_n \neq 0,$$

the series is a divergent one. On the other hand, if the condition is satisfied, the series is not necessarily convergent and may be divergent.

Each term of a series is a finite number and the sum of a finite number of such terms is also a finite number. Hence, the *convergence or divergence of any given series is unaffected by adding or discarding a finite number of terms.*

Exercise 69

GROUP A

Assuming that the formation of succeeding terms in each of the following series is as suggested by the given terms, write the general term for each.

1. $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$.
2. $1 + 3 + 5 + 7 + 9 + \cdots$.
3. $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$.
4. $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$.

5. $1 + \dfrac{2}{2} + \dfrac{3}{2^2} + \dfrac{4}{2^3} + \cdots.$

6. $\dfrac{1}{1 \cdot 3} + \dfrac{1}{3 \cdot 5} + \dfrac{1}{5 \cdot 7} + \cdots.$

7. $\dfrac{1}{1 \cdot 2} + \dfrac{1}{2 \cdot 3} + \dfrac{1}{3 \cdot 4} + \cdots.$

8. $\dfrac{1}{1 \cdot 3} + \dfrac{1}{5 \cdot 7} + \dfrac{1}{9 \cdot 11} + \cdots.$

Write the first four terms of each of the following series for which the general terms are as follows.

9. $\dfrac{2^n}{(n+1)!}.$ **11.** $(-1)^{n+1} \dfrac{2n}{n^2+1}.$

10. $\dfrac{n}{n^2+1}.$ **12.** $(-1)^{n-1} \dfrac{2^n}{n!}.$

Find the sum of each of the following series.

13. $1 + 4 + 7 + 10 + \cdots + 28.$

14. $2 + 4 + 8 + \cdots + 64.$

15. $1 + \tfrac{1}{2} + \tfrac{1}{4} + \cdots + \dfrac{1}{2^{n-1}} + \cdots.$

16. $1 + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \cdots.$

GROUP B

Assuming that the formation of succeeding terms in each of the following series is as suggested by the given terms, write the general term for each.

17. $\dfrac{1}{2} + \dfrac{1}{5} + \dfrac{1}{10} + \dfrac{1}{17} + \dfrac{1}{26} + \cdots.$

18. $\dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{5}} + \dfrac{1}{\sqrt{10}} + \dfrac{1}{\sqrt{17}} + \cdots.$

19. $\dfrac{1 \cdot 2}{3^3} + \dfrac{2 \cdot 3}{4^3} + \dfrac{3 \cdot 4}{5^3} + \dfrac{4 \cdot 5}{6^3} + \cdots.$

20. $\dfrac{1}{2!} + \dfrac{2}{3!} + \dfrac{3}{4!} + \dfrac{4}{5!} + \cdots.$

21. Derive the general expression for the sum of n terms of an arithmetic series whose first term is a and common difference is d. Find the expression for the last term.

22. Derive the general expression for the sum of n terms of a geometric series whose first term is a and common ratio is r. Find the expression for the last term.

Show that each of the following series is divergent.

23. $1 + 2 + 3 + 4 + 5 + \cdots.$

24. $1 + 2 + 4 + 8 + 16 + \cdots.$

25. $1 + 1 + 1 + 1 + 1 + \cdots.$

26. $1 - 1 + 1 - 1 + \cdots.$

27. Show that every infinite arithmetic series is divergent.

28. Write the general term for the series $1 + 2 + 4 + 8 + 16 + 32 + \cdots.$ Drop the first term of this series and write the general term. Drop the first two terms and write the general term. Continuing in this manner, show that the divergence of the series is unaffected by dropping a finite number of terms of the series.

101. Cauchy's Ratio Test.

One test of an infinite series for convergence, known as the *ratio test*, is due to A. L. Cauchy (1789–1857). Of all the tests, this is the most useful for infinite series which are commonly used. It is a test which is valid regardless of whether the terms of the series are all positive, or positive and negative. However, it fails to test certain series as we shall see. It makes use of the ratio of two successive terms of the series, hence its name.

Assume the infinite series of positive terms,

$$u_1 + u_2 + u_3 + \cdots + u_k + u_{k+1} + \cdots,$$

and let the ratio of the $(n+1)$st term and the nth term be denoted by

$$R(n) = \frac{u_{n+1}}{u_n}.$$

As indicated, this ratio is a function of n. In a given series, we seek the evaluation of the limit of this function as n increases without limit. The following are the criteria:

I.   If $\lim\limits_{n \to \infty} R(n) < 1$, *the series converges.*

II.   If $\lim\limits_{n \to \infty} R(n) > 1$, *the series diverges.*

III.   If $\lim\limits_{n \to \infty} R(n) = 1$, *the test fails.*

In Case III, some other test must be applied to the series. In other words, a series for which the limit of the function $R(n)$ is unity may be either convergent or divergent.

Proof for Case I.

$$\text{Let} \quad \lim_{n \to \infty} R(n) = L < r < 1.$$

Then the difference between $R(n)$ and L must become and remain as small as we please as n increases. Therefore, the ratio $R(n)$ for sufficiently large values of n, becomes and remains less than r. Let k represent such a value of n. From this, we may write the inequalities:

$$\frac{u_{k+1}}{u_k} < r, \qquad \text{or} \qquad u_{k+1} < r u_k.$$

$$\frac{u_{k+2}}{u_{k+1}} < r, \qquad \text{or} \qquad u_{k+2} < r u_{k+1} < r^2 u_k.$$

$$\frac{u_{k+3}}{u_{k+2}} < r, \qquad \text{or} \qquad u_{k+3} < r u_{k+2} < r^3 u_k.$$

$$\cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots$$

Adding corresponding members of these inequalities, we have

$$u_{k+1} + u_{k+2} + u_{k+3} + \cdots < u_k r(1 + r + r^2 + \cdots) = u_k r \left(\frac{1}{1-r}\right),$$

where $r < 1$. This shows that the sum of the series, neglecting the first k terms, is less than the last expression written above. This is finite since it is the product of three finite numbers. Therefore, the given series is convergent, as we may neglect a finite number of terms without affecting the convergence.

Proof for Case II.

$$\text{Let } \lim_{n \to \infty} R(n) = L > 1.$$

As n increases without limit, $R(n)$ becomes and remains greater than 1 for sufficiently large values of n. Let k represent such a value of n. From this, we may write the inequalities

$$\frac{u_{k+1}}{u_k} > 1, \qquad \text{or} \qquad u_{k+1} > u_k.$$

$$\frac{u_{k+2}}{u_{k+1}} > 1, \qquad \text{or} \qquad u_{k+2} > u_{k+1} > u_k.$$

$$\frac{u_{k+3}}{u_{k+2}} > 1, \qquad \text{or} \qquad u_{k+3} > u_{k+2} > u_k.$$

$$\cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots$$

Adding corresponding members of these inequalities, we have

$$u_{k+1} + u_{k+2} + u_{k+3} + \cdots > u_k + u_k + u_k + \cdots = (n - k)u_k.$$

This shows that the sum of the series, neglecting the first k terms, is greater than the last expression written above. But since

$$\lim_{n \to \infty} (n - k)u_k$$

does not exist, the given series is divergent.

Given the series

$$\frac{1}{2^2} + \frac{1}{2 \cdot 2^4} + \frac{1}{3 \cdot 2^6} + \cdots + \frac{1}{n \cdot 2^{2n}} + \cdots.$$

To test the convergence of this series, we apply the ratio test as follows:

$$\frac{u_{n+1}}{u_n} = \frac{n \, 2^{2n}}{(n+1) \, 2^{2n+2}} = \frac{n}{4(n+1)}.$$

Taking the limit,

$$\lim_{n \to \infty} \frac{n}{4(n + 1)} = \frac{1}{4}.$$

Hence, the given series is convergent.

Alternating Series. A series in which the terms are alternately positive and negative is called an *alternating series*. A series some of whose terms are negative is convergent if the series formed from the absolute values of the terms of the given series is convergent. Such a series is called an *absolutely convergent series*. However, there are alternating series which are convergent when the series of the absolute values of its terms is divergent. An example of such a series is

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} + \cdots,$$

which is convergent. The series which is formed from the absolute values of the terms of this series is shown in the next section to be a divergent series.

A simple convergence test for an alternating series,

$$u_1 - u_2 + u_3 - u_4 + \cdots,$$

where u_1 is positive, is as follows: If after a certain term,

$$u_n > u_{n+1} \quad \text{and} \quad \lim_{n \to \infty} u_n = 0,$$

the series is convergent. Using this test, the alternating series above is shown to be convergent.

Exercise 70

Determine whether each of the following series is convergent or divergent. In case the ratio test fails, apply the necessary, though not sufficient, condition that the limit of the general term be zero.

1. $\dfrac{1}{2} + \dfrac{2}{2^2} + \dfrac{3}{2^3} + \cdots.$

2. $1 + \dfrac{1}{2!} + \dfrac{1}{3!} + \dfrac{1}{4!} + \cdots.$

3. $\dfrac{1}{2} + \dfrac{1}{2 \cdot 2^2} + \dfrac{1}{3 \cdot 2^3} + \cdots.$

4. $1 + 4 + 9 + 16 + \cdots.$

5. $1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \cdots.$

6. $1 + \dfrac{2}{3} + \dfrac{3}{3^2} + \dfrac{4}{3^3} + \cdots.$

7. $1 + \dfrac{1}{3!} + \dfrac{1}{5!} + \dfrac{1}{7!} + \cdots$.

8. $\dfrac{2}{2!} - \dfrac{3}{4!} + \dfrac{4}{6!} - \dfrac{5}{8!} + \cdots$.

9. $1 + \dfrac{2^2}{2!} + \dfrac{3^2}{3!} + \dfrac{4^2}{4!} + \cdots$.

10. $\frac{1}{3} + \frac{2}{5} + \frac{3}{7} + \frac{4}{9} + \cdots$.

11. $3 + \dfrac{3^2}{2!} + \dfrac{3^3}{3!} + \dfrac{3^4}{4!} + \cdots$.

12. $2 + \dfrac{2^2}{2} + \dfrac{2^3}{3} + \dfrac{2^4}{4} + \cdots$;

102. Integral Test.

In this section the convergence or divergence of the series

$$1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \frac{1}{5^p} + \cdots + \frac{1}{n^p} + \cdots$$

is considered for various real values of p. This series is an important one for our purposes as it is used in the comparison test given in the section following.

If the ratio test is applied to this series for any integral value of p, it fails since

$$\lim_{n \to \infty} \frac{n^p}{(n+1)^p} = \lim_{n \to \infty} \left[\frac{n}{(n+1)} \right]^p = 1.$$

We consider the four following cases:

(1) If $p < 0$.   (2) If $p = 1$.   (3) If $0 < p < 1$.   (4) If $p > 1$.

(1). **If $p < 0$,** the limit of the general term is taken. Let $|p| = q$. Then

$$\lim_{n \to \infty} \frac{1}{n^p} = \lim_{n \to \infty} n^q = \infty,$$

and the series is *divergent*.

(2). **If $p = 1$,** the resulting series is known as the *harmonic series*,

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{n} + \cdots.$$

In order to study the convergence or divergence of this series, let us draw the curve $y = 1/x$ for positive values of x, as in Figure 83. The area under the curve from $x = 1$ to $x = b$ as b becomes infinite, is expressed by

the limit of the definite integral,

$$\lim_{b \to \infty} \left[\int_1^b \frac{dx}{x} \right] = \lim_{b \to \infty} \left[\ln x \right]_1^b = \infty.$$

Hence this limit does not exist. Rectangles having bases each equal to 1 are circumscribed to the area under the curve from $x = 1$ to $x = b$ as shown in the figure. The area of each rectangle is equal to the corresponding term in the given series. That is, the area of the first rectangle is 1, the second is $\frac{1}{2}$, the third is $\frac{1}{3}$, etc. The area of the first n rectangles is *greater* than the area under the curve from $x = 1$ to $x = n + 1$. For example, if $n = 4$,

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > \int_1^5 \frac{dx}{x},$$

or, $\quad \dfrac{25}{12} > \ln 5, \qquad 2.083 > 1.609.$

Thus,

$$\lim_{n \to \infty} S_n \geqq \lim_{n \to \infty} \left[\int_1^{n+1} \frac{dx}{x} \right] = \infty.$$
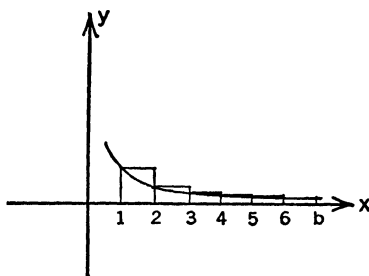
Fig. 83

Hence, the given series is divergent.

(3). **If $0 < p < 1$,** we compare the terms of the series with the area under a curve.

In order to study the convergence or the divergence of this series, consider the function

$$y = \frac{1}{x^p}$$

and draw the curve similar to the one drawn in **Figure 83.** The area under the curve from $x = 1$ to $x = b$ as b becomes infinite, is expressed by the limit of the definite integral,

$$\lim_{b \to \infty} \left[\int_1^b \frac{dx}{x^p} \right] = \lim_{b \to \infty} \left[\frac{x^{1-p}}{1-p} \right]_1^b$$

$$= \lim_{b \to \infty} \left[\frac{b^{1-p} - 1}{1 - p} \right] = \infty.$$

But since $p < 1$ makes the exponent of b positive, this limit does not exist. As in the previous case, rectangles having bases each equal to 1 are circumscribed to the area under the curve from $x = 1$ to $x = b$, as shown in the

figure. The area of each rectangle is equal to the corresponding term in the given series. The combined area of the first n rectangles is greater than the area under the curve from $x = 1$ to $x = n + 1$. Thus,

$$\lim_{n\to\infty} S_n \geq \lim_{n\to\infty} \left[\int_1^{1+n} \frac{dx}{x^p} \right] = \infty.$$

Hence, the given series is divergent for any value of p in the interval $0 < p < 1$.

(4). **If $p > 1$,** the character of the series is changed.

Consider the function

$$y = \frac{1}{x^p}$$

and draw the curve as in Figure 84. The area under the curve from $x = 1$ to $x = b$ as b becomes infinite, is expressed by the limit of the definite integral,
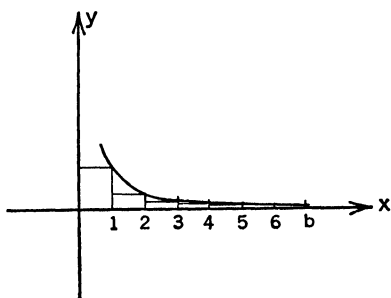
$$\lim_{b\to\infty} \left[\int_1^b \frac{dx}{x^p} \right] = \lim_{b\to\infty} \left[\frac{x^{1-p}}{1-p} \right]_1^b$$

$$= \lim_{b\to\infty} \left[\frac{\dfrac{1}{b^{p-1}} - 1}{1-p} \right] = \frac{1}{p-1}.$$

FIG. 84

Rectangles having bases each equal to 1 are inscribed to the area under the curve from $x = 0$ to $x = b$ as shown in the figure. The area of each rectangle is equal to the corresponding term in the given series. That is, the area of the first rectangle is 1, the second is $1/2^p$, the third is $1/3^p$, etc. The area under the curve from $x = 1$ to $x = n$ is greater than the area of the $(n - 1)$ inscribed rectangles, omitting the first. Thus,

$$S_n - 1 < \int_1^n \frac{dx}{x^p},$$

$$S - 1 = \lim_{n\to\infty} (S_n - 1) \leq \lim_{n\to\infty} \left[\int_1^n \frac{dx}{x^p} \right] = \frac{1}{p-1},$$

from which

$$S \leq \frac{p}{p-1}.$$

Hence the given series is convergent.

103. Comparison Test.

The *comparison test* of a series is one in which the terms of a given series are compared with those of series known to be convergent or divergent.

Let the infinite series

(c) $$c_1 + c_2 + c_3 + \cdots + c_n + \cdots$$

be one which is known to converge, let the infinite series

(d) $$d_1 + d_2 + d_3 + \cdots + d_n + \cdots$$

be one which is known to diverge and let the infinite series of positive terms

(u) $$u_1 + u_2 + u_3 + \cdots + u_n + \cdots$$

be one whose convergence or divergence is to be established. The series (u) is convergent if, after some term, each of its terms is less than or at most equal to, the corresponding term of the series (c), known to converge. Thus, if

$$\lim_{n \to \infty} \sum_{i=1}^{i=n} c_i = C, \qquad \sum_{i=1}^{i=n} u_i \leqq \sum_{i=1}^{i=n} c_i < C.$$

Then,

$$\lim_{n \to \infty} \sum_{i=1}^{i=n} u_i \leqq \lim_{n \to \infty} \sum_{i=1}^{i=n} c_i = C$$

and the series (u) is convergent.

Similarly, the series (u) is divergent if, after some term, each of its terms is greater than or at least equal to, the corresponding term of the series (d), known to diverge. Thus,

if $$\lim_{n \to \infty} \sum_{i=1}^{i=n} d_i = \infty, \qquad \sum_{i=1}^{i=n} u_i \geqq \sum_{i=1}^{i=n} d_i = \infty.$$

Then, $$\lim_{n \to \infty} \sum_{i=1}^{i=n} u_i = \infty,$$

and the series (u) is divergent.

Consider the series

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots + \frac{1}{(2n-1)^2} + \cdots.$$

The ratio test fails, since

$$\lim_{n \to \infty} \frac{(2n-1)^2}{(2n+1)^2} = 1.$$

In application of the comparison test, we choose the series

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots + \frac{1}{n^2} + \cdots,$$

which is known to converge from (4), Section 102. Comparing the series, the first terms are equal. Thereafter, each term of the given series is less than the corresponding term of the latter. Hence, the given series converges.

Consider the series

$$\frac{1}{\ln 2} + \frac{1}{\ln 3} + \frac{1}{\ln 4} + \cdots + \frac{1}{\ln (n + 1)} + \cdots.$$

For comparison, we use the harmonic series (2), Section 102, which is known to diverge. Omitting the first term of the harmonic series, each of its terms is less than the corresponding term of the given series. Hence, the given series is divergent.

Exercise 71

Establish the convergence or the divergence of each of the following series by comparison with either the p-series or other series known to converge or diverge.

1. $\dfrac{1}{4} + \dfrac{1}{6} + \dfrac{1}{10} + \cdots + \dfrac{1}{2^n + 2} + \cdots.$

2. $\dfrac{1}{2} + \dfrac{2}{5} + \dfrac{3}{10} + \cdots + \dfrac{n}{n^2 + 1} + \cdots.$

3. $1 + \dfrac{3}{9} + \dfrac{3}{19} + \cdots + \dfrac{3}{2n^2 + 1} + \cdots.$

4. $1 + \dfrac{1}{2^3} + \dfrac{1}{3^3} + \cdots + \dfrac{1}{n^3} + \cdots.$

5. $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \cdots.$

6. $\frac{1}{3} + \frac{2}{5} + \frac{3}{7} + \frac{4}{9} + \cdots.$

7. $\dfrac{1}{2 \cdot 2} - \dfrac{2}{2^2 \cdot 5} + \dfrac{3}{2^3 \cdot 10} - \dfrac{4}{2^4 \cdot 17} + \cdots.$

8. $\frac{1}{3} + \frac{1}{1^5} + \frac{1}{3^5} + \frac{1}{6^3} + \cdots.$

9. $\dfrac{2}{1 \cdot 2} + \dfrac{2}{3 \cdot 4} + \dfrac{2}{5 \cdot 6} + \dfrac{2}{7 \cdot 8} + \cdots.$

10. $1 - \dfrac{1}{3!} + \dfrac{1}{5!} - \dfrac{1}{7!} + \cdots.$

11. It was assumed in the last illustration above, that the logarithm of a number is less than that number. Prove that this is true.

104. Power Series.

A series which assumes the form

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots$$

in which x is a variable and in which the coefficients are constants, is known as a *power series*. The remainder of the chapter is devoted to a study of such series.

A power series may converge for all values of the variable x, or it may not converge for any values of x, save $x = 0$. However, it is usual for a power series to converge for all values of x within some finite interval and to diverge for all others outside that interval.

Interval of Convergence. The totality of values of the variable for which a given power series converges is called the *interval of convergence* of the series. In the applications it is of fundamental importance to be able to find the interval of convergence for a given power series.

For most power series the ratio test may be used to find the interval of convergence. For example, given the series

$$1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{x^n}{n} + \cdots.$$

Taking the limit of the absolute value of the ratio of the $(n + 1)$st term and the nth term as n becomes infinite, we have

$$\lim_{n \to \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \to \infty} \left| \frac{n x^{n+1}}{(n + 1) x^n} \right| = \lim_{n \to \infty} \frac{n}{n + 1} \cdot |x| = |x|.$$

Therefore, by the criteria given in Section 101:

if $|x| < 1$, the series converges for $-1 < x < 1$,

if $|x| > 1$, the series diverges for $x < -1$ and $x > 1$,

if $|x| = 1$, the test fails.

If $x = 1$, the series is the harmonic series which is known to be divergent. If $x = -1$, the series is the alternating series given in Section 101, where it is said to be convergent. Hence, the given series has the interval of convergence,

$$-1 \leqq x < 1.$$

As shown in this illustration, the interval of convergence of a power series may be found by the application of the ratio test, but the behavior of the series at the *end points* of the interval must be tested by some other criteria.

Exercise 72

Find the interval of convergence for each of the following series, testing each of the end points.

1. $1 + \dfrac{x}{2} + \dfrac{x^2}{3} + \dfrac{x^3}{4} + \cdots .$

2. $1 + 2x + 3x^2 + 4x^3 + \cdots .$

3. $1 + \dfrac{x}{3} + \dfrac{x^2}{5} + \dfrac{x^3}{7} + \cdots .$

4. $1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \cdots .$

5. $1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^2}{6!} + \cdots .$

6. $1 - 3x + 5x^2 + 7x^3 + \cdots .$

7. $x - \dfrac{x^2}{2} + \dfrac{x^3}{3} + \dfrac{x^4}{4} + \cdots .$

8. $1 + \dfrac{x}{2} + \dfrac{2^2 x^2}{2^2} + \dfrac{3^2 x^3}{2^3} + \dfrac{4^2 x^4}{2^4} + \cdots .$

9. $1 + \dfrac{1}{x} + \dfrac{1}{2x^2} + \dfrac{1}{3x^3} + \cdots .$

10. $\dfrac{1}{x} + \dfrac{1 \cdot 3}{x^2} + \dfrac{1 \cdot 3 \cdot 5}{x^3} + \dfrac{1 \cdot 3 \cdot 5 \cdot 7}{x^4} + \cdots .$

11. $\dfrac{1}{2} + \dfrac{2(x-2)}{3} + \dfrac{3(x-2)^2}{4} + \cdots .$

12. $\dfrac{(x+2)}{4} + \dfrac{2(x+2)^2}{4^2} + \dfrac{3(x+2)^3}{4^3} + \cdots .$

105. Maclaurin's Series.

A power series which is convergent for $a < x < b$, has a sum which is a function of x, valid for only those values of x in the interval of convergence. Consider the series obtained by the indicated division,

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots .$$

Since,

$$\lim_{n \to \infty} \left| \frac{u_{n+1}}{u_n} \right| = |x|,$$

the interval of convergence of the series is $-1 < x < 1$. The series is divergent at each of the end points. Also, the given function is not defined for $x = 1$ and the series does not represent the function for $x = -1$, since

$$\tfrac{1}{2} \neq 1 - 1 + 1 - 1 \cdots .$$

Similarly, the series does not represent the function for values of $x < -1$ or for values of $x > 1$. If, however, x has any value in the interval of convergence, the infinite power series is said to *represent* the function.

This discussion suggests an important problem, that of expressing a given function as an infinite power series. This process is known as *expanding a function in a power series*. While it is by no means possible to expand all functions in a power series, it is now our task to represent certain functions by such series within the interval of convergence.

Let us assume that sin x can be expanded in a power series in x, in radians, and write

$$\sin x = A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + \cdots,$$

in which A, B, C, $\cdots$ are constants to be determined. The successive derivatives with respect to x are as follows:

$$\cos x = B + 2Cx + 3Dx^2 + 4Ex^3 + 5Fx^4 + \cdots$$
$$-\sin x = 2C + 6Dx + 12Ex^2 + 20Fx^3 + \cdots$$
$$-\cos x = 6D + 24Ex + 60Fx^2 + \cdots$$
$$\sin x = 24E + 120Fx + \cdots$$
$$\cos x = 120F + \cdots$$

in which it has been assumed that series may be differentiated term by term. Making the assumption that the original equality and those derived from it, hold for $x = 0$, the values of the constants can be found. Since sin $0 = 0$ and cos $0 = 1$, we have

$$A = 0, \quad B = 1, \quad C = 0, \quad 6D = -1, \quad E = 0, \quad 120F = 1, \cdots$$

or

$$B = 1, \quad D = -\frac{1}{3!}, \quad F = \frac{1}{5!}, \quad H = \frac{1}{7!}, \cdots.$$

Replacing the constants in the original equation by these values, we have the series

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots.$$

This power series converges for all values of x. We say that it represents the function sin x for all values of x.

Let $f(x)$ be a function having derivatives of all orders, each of which can be evaluated for $x = 0$. Also, assume that $f(x)$ can be represented by

a power series convergent within an interval containing $x = 0$. Then the series and the derivatives are written as follows:

$$f(x) = A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + \cdots$$
$$f'(x) = B + 2Cx + 3Dx^2 + 4Ex^3 + 5Fx^4 + \cdots$$
$$f''(x) = 2C + 6Dx + 12Ex^2 + 20Fx^3 + \cdots$$
$$f'''(x) = 6D + 24Ex + 60Fx^2 + \cdots$$
$$f^{\mathrm{iv}}(x) = 24E + 120Fx + \cdots$$

If $x = 0$, the values of the constants are

$$A = f(0), \quad B = f'(0), \quad C = \frac{f''(0)}{2!}, \quad D = \frac{f'''(0)}{3!}, \cdots.$$

Substituting these values for the coefficients,

$$f(x) = f(0) + f'(0)\, x + \frac{f''(0)}{2!}\, x^2 + \frac{f'''(0)}{3!}\, x^3 + \cdots.$$

This series is called a *Maclaurin's series.*

Expanding a function in a Maclaurin's series is also known as *developing* a function $f(x)$ in powers of x in the *neighborhood of zero.* A function cannot be developed by the Maclaurin expansion unless the function and its derivatives exist for $x = 0$. In case it can be expanded, the series represents the function for all values of x in the interval of convergence. However, as we shall see in the applications, a Maclaurin's series is most useful in finding values of $f(x)$ for values of x near zero.

To expand the function e^x in a Maclaurin's series, we proceed as follows:

Let $\qquad\qquad f(x) = e^x, \quad$ then $\quad f(0) = 1.$

Also, $\qquad\qquad f'(x) = e^x, \qquad\qquad f'(0) = 1,$

$\qquad\qquad\qquad f''(x) = e^x, \qquad\qquad f''(0) = 1,$

Hence,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots,$$

which is a series convergent for all values of x.

Sometimes the Maclaurin expansion of a function can be simplified by a transformation of the variable. For example, the expansion of the func-

tion $\sin x^2$ may be obtained by letting $x^2 = z$. Then,

$$\sin x^2 = \sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots,$$

as developed above. Making the reverse substitution,

$$\sin x^2 = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} + \cdots.$$

Exercise 73

GROUP A

Verify each of the following important series expansions and their intervals of convergence.

1. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots.$ All values of x.

2. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots.$ All values of x.

3. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots.$ All values of x.

4. $\ln (x + 1) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots.$ $-1 < x \leqq 1.$

5. $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots.$ $-1 \leqq x < 1.$

Expand each of the following functions into a Maclaurin's series, obtaining the first four nonvanishing terms.

6. $\arcsin x.$

7. $\ln (a + x).$

8. $\sin \left(\frac{\pi}{4} + x \right).$

9. $\sqrt{1 + x^2}.$

10. $\cos ax.$

11. $\frac{1}{2}(e^x + e^{-x}).$

12. $a^x.$

13. $\ln (1 + \sin x).$

GROUP B

Expand each of the following functions into a Maclaurin's series, obtaining the first four nonvanishing terms.

14. $\ln \cos x.$

15. $\cos^2 x.$

16. $(e^x + e^{-x})^2.$

17. $\sin \left(\frac{\pi}{6} - x \right).$

18. $\sqrt[3]{(1 - x)^2}.$

19. $\sec x.$

20. $e^{\sin x}.$

21. $\ln (x + \sqrt{1 + x^2}).$

22. $\frac{1}{\sqrt{4 - x}}.$

23. $x \arctan x.$

24. $e^{\tan x}.$

25. $\cos x^2.$

26. $e^{x^2}.$

27. $\frac{1}{\sqrt{1 - x^4}}.$

106. Computation by a Maclaurin Expansion.

One of the most important elementary applications of the expansion of a function into a series is the computation of various values of the function.

To compute sin 2° to five decimal places, we use the expansion of sin x, where

$$x = 2° = \frac{\pi}{90} \text{ radians.}$$

$$\sin 2° = \frac{\pi}{90} - \frac{1}{6}\left(\frac{\pi}{90}\right)^3 + \frac{1}{120}\left(\frac{\pi}{90}\right)^5 - \cdots$$

$$= 0.034907 - 0.000007 = 0.03490$$

This series converges rapidly, so that in the computation two terms are sufficient. It is obvious, without computing, that the third term is too small to affect the sixth decimal place.

To compute sin 62° to five decimal places, the series for sin x converges too slowly to be practical, where

$$x = 62° = \frac{31}{90}\pi \text{ radians.}$$

Consequently, the series for sin $(x + \pi/3)$ is used, where

$$x = 2° = \frac{\pi}{90} \text{ radians.}$$

$$\sin\left(x + \frac{\pi}{3}\right) = \frac{1}{2}\left(\sqrt{3} + x - \frac{\sqrt{3}}{2!}x^2 - \frac{1}{3!}x^3 + \frac{\sqrt{3}}{4!}x^4 + \cdots\right).$$

Replacing x by its value,

$$\sin 62° = \tfrac{1}{2}(1.73205 + 0.03491 - 0.00105 - 0.00001) = 0.88295.$$

Maximum Error. In an alternating series, the absolute value of the maximum error committed by stopping with any particular term of the series is less than, or equal to, the absolute value of the term succeeding the last term used in the computation. This is assumed without proof.

In the sin x series, if we use

$$\sin x = x,$$

$$|\text{ error }| \leqq \left|\frac{x^3}{6}\right|.$$

While if we use

$$\sin x = x - \frac{x^3}{3\,!},$$

$$\text{error} \mid \leqq \left|\frac{x^5}{5\,!}\right|.$$

If a maximum error of 0.0005 is allowed in the computation of the sine of an angle, we can find the limits within which the first term only, may be used.

$$\frac{x^3}{6} = 0.0005, \quad x^3 = 0.003$$

$$x = 0.1442 \text{ radians} = 8° \ 15', \text{ approximately.}$$

With the same allowable error, we find the limits within which the first two terms may be used.

$$\frac{x^5}{120} = 0.0005, \quad x^5 = 0.06$$

$$x = 0.5697 \text{ radian} = 32° \ 39', \text{ approximately.}$$

Differentiation and Integration of Series. If a function is represented by a power series, the derivative of the function is represented by the series obtained by the term by term differentiation of the given series. The series obtained is valid for those values of the variable within the interval of convergence. Similarly, and under the same conditions, the integration of a function is represented by the series obtained by the term by term integration of the given series.

Consider the function

$$f(x) = \frac{\sin x}{x}.$$

While $f'(x)$ can be obtained readily by the usual methods, the $\int f(x)\,dx$ cannot be obtained by such methods. In the solutions of the two problems which follow, use is made of the differentiation and the integration of series.

To find the slopes of the given curve at the points $x = 0$ and $x = 1$, we use

$$f(x) = 1 - \frac{x^2}{3\,!} + \frac{x^4}{5\,!} - \frac{x^6}{7\,!} + \cdots$$

$$f'(x) = -\frac{2}{3\,!}x + \frac{4}{5\,!}x^3 - \frac{6}{7\,!}x^5 + \cdots.$$

At $x = 0$, $f'(0) = 0$, and the curve has a horizontal tangent.

At $x = 1$, $f'(1) = -0.33333 + 0.03333 - 0.00119 = -0.30119$.

The absolute value of the error, using the first neglected term, is

$$| \text{ error } | \leqq \left| \frac{8}{9!} x^7 \right| = 0.00002.$$

The area under the curve $y = \dfrac{\sin x}{x}$ from $x = 0$ to $x = 1$ may be found as follows:

$$
\begin{aligned}
S &= \int_0^1 \frac{\sin x}{x}\, dx = \int_0^1 \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots \right) dx \\
&= x - \frac{x^3}{18} + \frac{x^5}{600} - \frac{x^7}{35280} + \cdots \Bigg]_0^1 \\
&= 1 - 0.05556 + 0.00167 = 0.9461,
\end{aligned}
$$

in which the absolute value of the error is less than 0.00003.

Exercise 74

GROUP A

1. Compute the value of e to four decimal places.
2. Compute the value of $\cos 2°$ to four decimal places.
3. Compute the value of $\sin 3°$ to four decimal places.
4. Compute $e^{1\,2}$ to four decimal places.
5. Compute ln (1.2) to four decimal places.
6. Compute $\sqrt{e}$ to four decimal places.

GROUP B

7. Expand $\ln \dfrac{1+x}{1-x}$ into a Maclaurin's series.
8. Using the expansion in Problem 7, find ln 2 to four decimal places.
9. From the expansion of arcsin x and arcsin $\frac{1}{2} = \pi/6$, compute the value of π to three decimal places.
10. Expand $\sqrt{9 + x}$ and compute $\sqrt{9.02}$ to five decimal places.
11. Compute sin 32° to five decimal places.
12. Compute cos 62° to five decimal places.
13. Show that cot x cannot be expanded in a power series in x.
14. From the expansion of arctan x and arctan $\frac{1}{7} + 2$ arctan $\frac{1}{3} = \pi/4$, compute the value of π to four decimal places.
15. Compute ln sec 46° to four decimal places.
16. Within what interval can sin x be replaced by x, if the allowable error is 0.0001?
17. Within what interval can cos x be replaced by $1 - \frac{1}{2}x^2$, if the allowable error is 0.0001?

18. Expand $e^{-x^2/2}$ into a Maclaurin's series.

19. Using the series obtained in Problem 18, evaluate $\int_0^{0.2} e^{-x^2/2}\,dx$ to four decimal places.

20. Evaluate $\lim_{x\to 0} \dfrac{\sin x}{x}$ by means of series.

21. Evaluate $\lim_{x\to 0} \dfrac{\arcsin x}{x}$ by means of series.

22. Evaluate $\int_0^{0.5} \sqrt{1 - x^3}\,dx$ to four decimal places.

23. Evaluate $\lim_{x\to 0} \dfrac{1 - \cos x}{x}$ by means of series.

24. Evaluate $\lim_{x\to 0} \dfrac{e^x - e^{-x}}{2x}$ by means of series.

25. Find the area under the curve $y = e^{-x^2}$ from $x = 0$ to $x = 1$, to four decimal places.

107. Taylor's Series.

It is often desirable to expand a function into a power series of a binomial $(x - a)$ instead of a power series in x for reasons which are given in the next section. In expanding a function in a series of $(x - a)$, the constant a is some given number or is chosen at pleasure. The procedure is analogous to that of the Maclaurin expansion.

Let $f(x)$ be a function whose successive derivatives exist, each of which can be evaluated for $x = a$. Also, let

$$f(x) = A + B(x - a) + C(x - a)^2 + D(x - a)^3 + \cdots.$$

By differentiation,

$$f'(x) = B + 2C(x - a) + 3D(x - a)^2 + \cdots$$
$$f''(x) = 2C + 6D(x - a) + \cdots$$
$$f'''(x) = 6D + \cdots$$

$$\cdot \quad \cdot \quad \cdot \cdot \quad \cdot \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

By letting $x = a$, the values of the constants are determined,

$$A = f(a), \quad B = f'(a), \quad C = \frac{f''(a)}{2!}, \quad D = \frac{f'''(a)}{3!}, \cdots$$

Substituting values,

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \frac{f'''(a)}{3!}(x + a)^3 + \cdots.$$

This series is known as *Taylor's series.*

Let us expand $\ln x$ in a power series of $(x - 1)$. This is a Taylor expansion in which $a = 1$.

If $\qquad\qquad f(x) \;\; = \ln x, \qquad f(1) \;\; = 0.$

Also $\qquad\quad f'(x) \;\; = x^{-1}, \qquad f'(1) \;\; = 1,$

$\qquad\qquad\qquad f''(x) \;\; = -x^{-2}, \qquad f''(1) \;\; = -1,$

$\qquad\qquad\qquad f'''(x) = 2x^{-3}, \qquad f'''(1) = 2.$

$\cdots\cdots \quad \cdot \quad \cdots\cdots\cdots \quad \cdot \quad \cdots \quad \cdots$

Hence,

$$\ln x = (x - 1) - \tfrac{1}{2}(x - 1)^2 + \tfrac{1}{3}(x - 1)^3 - \tfrac{1}{4}(x - 1)^4 + \cdots .$$

Using the ratio test for this series,

$$\lim_{n \to \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \to \infty} \left| \frac{n}{n + 1}\, (x - 1) \right| = |\, x - 1 \,| .$$

Consequently, the series converges for $|\, x - 1 \,| < 1$, or for $0 < x \leqq 2$.

Exercise 75

GROUP A

Expand each of the following functions into a Taylor's series, obtaining the first four terms and using the values of a given.

1. e^x for $a = 4$.
2. $\sin x$ for $a = \pi/6$.
3. $\cos x$ for $a = \pi/3$.
4. $\arctan x$ for $a = 1$.

5. $\tan x$ for $a = \pi/4$.
6. e^{2x} for $a = 2$.
7. $\sqrt{1 + x^2}$ for $a = 1$.
8. $\sqrt{x}$ for $a = 9$.

GROUP B

9. Expand $\ln x$ about $x = 2$.
10. Expand $\sqrt[3]{x}$ in a power series in $(x - 8)$.
11. Expand $x^3 - 2x^2 - x - 5$ in powers of $(x - 1)$.
12. Using the result of Problem 11, evaluate $\displaystyle\int \frac{x^3 - 2x^2 - x - 5}{(x - 1)^4}\, dx$.
13. Expand $\ln (\sin x)$ about $x = \pi/4$.
14. Find $\displaystyle\int_0^a \frac{dx}{\sqrt{1 - x^3}}$ by means of a series, where $a < 1$.

Verify each of the following series and their intervals of convergence.

15. $(x + a)^k = a^k + ka^{k-1} x + \dfrac{k(k - 1)}{2\,!}\, a^{k-2}\, x^2 + \cdots$

$\qquad + \dfrac{k(k - 1)(k - 2) \cdots (k - n + 2)}{(n - 1)\,!}\, a^{k-n+1}\, x^{n-1} + \cdots .$

If k is zero or a positive integer, the series is finite and holds for all values of x. For other real values of k, the series is infinite and holds for $-a < x < a$.

16. $\ln x = \ln a + \dfrac{1}{a}(x - a) + \dfrac{1}{2a^2}(x - a)^2 + \cdots$, $a > 0$, for values of x,

$0 < x \leqq 2a$. Write the nth term.

17. $e^x = e^a \left[1 + (x - a) + \dfrac{(x - a)^2}{2!} + \cdots \right]$, for all values of x Write

the nth term.

18. $\sin x = \sin a + (x - a) \cos a - \dfrac{(x - a)^2}{2!} \sin a - \dfrac{(x - a)^3}{3!} \cos a + \cdots$,

for all values of x.

19. $\cos x = \cos a - (x - a) \sin a - \dfrac{(x - a)^2}{2!} \cos a + \dfrac{(x - a)^3}{3!} \cos a + \cdots$,

for all values of x.

108. Computation by a Taylor Expansion.

As has been stated, one of the most important applications of series is in the computation of values of a function for different values of the variable. The tables of values of the logarithms of numbers and of the trigonometric functions of angles may be computed in this way. For this purpose it is necessary that the series used for the computation, converge. For practical purposes, it is desirable that the series used converge rapidly. It should be possible to find a series to represent a function so that for the value sought, the first few terms are sufficient.

If a Maclaurin expansion is to be used, the rapidity of the convergence, assuming that it does converge, depends on the numerical values of x and the coefficients of the series. In general, the smaller x is, the more rapidly the terms of the series diminish in numerical value. Hence, if $f(x)$ is to be computed for a given *small* value of x and if $f(0)$, $f'(0)$, $f''(0)$, $\cdots$ can be computed easily, a Maclaurin series usually gives an approximation of the value of $f(x)$. Under such circumstances, we say that a Maclaurin series is applicable in the *neighborhood of the origin*.

If a Taylor expansion is to be used, the rapidity of the convergence, assuming that it does converge, depends on the numerical values of $(x - a)$ and the coefficients of the series. In general, the smaller $(x - a)$ is, the more rapidly the terms of the series diminish in numerical value. Hence, if $f(x)$ is to be computed for a given value of x and if a value of a can be chosen so that $(x - a)$ is *small* and $f(a)$, $f'(a)$, $f''(a)$, $\cdots$ can be computed easily, a Taylor series usually gives an approximation of the value of $f(x)$. Under such circumstances, we say that a Taylor series is applicable in the *neighborhood of $x = a$*.

To compute $\sin 62°$ we may expand $\sin x$ in a Taylor's series about $x = \pi/3$.

$$f(x) = \sin x, \qquad f\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}.$$

$$f'(x) = \cos x, \qquad f'\left(\frac{\pi}{3}\right) = \frac{1}{2}.$$

$$f''(x) = -\sin x, \qquad f''\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}.$$

$$\sin x = \frac{1}{2}\left[\sqrt{3} + \left(x - \frac{\pi}{3}\right) - \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right)^2 + \cdots\right].$$

When $x = 62° = \frac{31}{90}\pi$, the same series results as was obtained in Section 106, where it was found that $\sin 62° = 0.88295$.

Exercise 76

GROUP A

Compute each of the following by series to four decimal places.

1. $\sin 61°$.
2. $\cos 62°$.
3. $\sin 35°$.
4. $\ln \frac{3}{2}$.
5. $\sqrt[4]{15}$.
6. $e^{-0.2}$.

GROUP B

7. Compute $\cos 31°$ by expansion about $x = \pi/6$ using four terms. Show that the result is correct to three decimal places, since the error is 0 00006.

8. To find $\cos 66°$ to four decimal places, what series and how many terms should be used?

9. Using the series for $\ln x$ about $x = 4$ to obtain an approximation for $\ln 4.2$, find the maximum error if 5 terms are used. If the decimal is to be correct to four decimal places, how many terms of the expansion are necessary?

10. Find $\sqrt[3]{(7.95)^2}$ by a Taylor's series for $\sqrt[3]{x^2}$ to five decimal places.

11. Find $\sqrt[4]{16.5}$ to five decimal places.

12. Find the volume generated by the rotation of the area bounded by the curve $y = \frac{1}{x}\sin\frac{x}{2}$, the x-axis, the y-axis and the line $x - 1 = 0$, correct to four decimal places.

CHAPTER XIII

POLAR COORDINATES

109. Polar Coordinates.

The polar coordinate system consists of a fixed point, the *pole*, and a fixed line, the *polar axis*, through the pole. The polar coordinates (ρ, θ) of a point are the *radius vector* and the *vectorial angle*, respectively. The vectorial angle is measured from the polar axis and may be positive or negative according as it is measured in the counterclockwise or the clockwise direction. The radius vector is measured from the pole and may be positive or negative according as it is taken along the terminal side of the angle or along the terminal side produced through the pole.

The polar coordinate system is well adapted to the expression of the equation of a curve which involves a motion about a point. The equations of some curves are seldom expressed in any other system than the polar, others are usually expressed in the rectangular system, while still others may be equally well expressed in either system. In this chapter the methods of the calculus are applied to the equations of curves which are better expressed in a polar coordinate system.

The equation of a curve in polar coordinates usually assumes the form

$$\rho = f(\theta).$$

To plot the graph of such a function, a table of corresponding values of ρ and θ is prepared from which a smooth curve is drawn through the points located in a polar system. For this purpose it is convenient to use polar coordinate paper which

FIG. 85

consists of a series of concentric circles drawn about the pole and a number of radial lines through the pole.

The graph of the equation

$$\rho = 2 - 3 \cos \theta$$
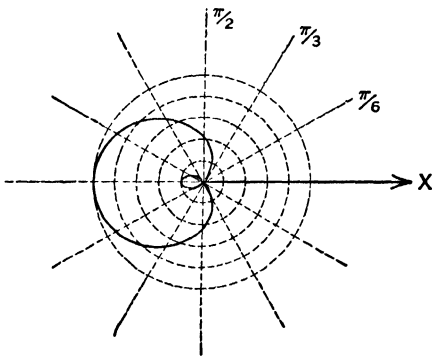
is drawn in Figure 85. It is to be observed that the curve is symmetrical with respect to the polar axis, since $\cos \theta = \cos (-\theta)$. The value of ρ is zero for the angles $\theta = \pm$ arccos $\frac{2}{3}$. This curve is known as the *limaçon*.

FIG. 86
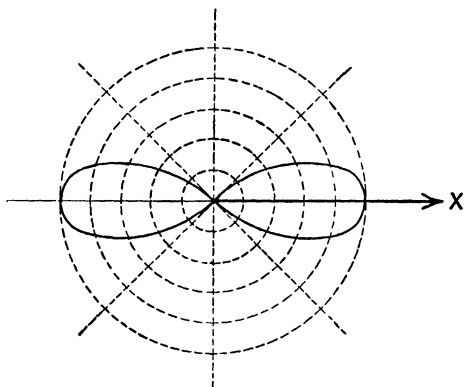
The graph of the curve known as the *lemniscate* is drawn in Figure 86 from the equation

$$\rho^2 = 16 \cos 2\theta.$$

The values of ρ are real for the following intervals of the angle,

$$-\frac{\pi}{4} \leqq \theta \leqq \frac{\pi}{4}, \quad \frac{3}{4}\pi \leqq \theta \leqq \frac{5}{4}\pi.$$

Exercise 77

GROUP A

Draw the graph of each of the following equations.

1. $\rho \cos \theta = 4$.
2. $\rho \sin \theta = 3$.
3. $\rho \cos \theta + 2 = 0$.
4. $\rho \sin \theta + 6 = 0$.
5. $\rho = 4 \cos \theta$.
6. $\rho = 2(1 - \cos \theta)$.

7. $\rho(1 + \cos \theta) = 4$.
8. $\rho(2 + \sin \theta) = 6$.
9. $\rho(1 - \cos \theta) = 2$.
10. $\rho(1 - 2 \cos \theta) = 4$.
11. $\rho = 4 \sin 2\theta$.
12. $\rho = 4 \cos 2\theta$.

Transform each of the following equations to polar coordinates or to rectangular coordinates.

13. $xy = 2$.
14. $x^2 = 4py$.
15. $x^2 + y^2 = a^2$.

16. $\rho = 2a \sin \theta$.
17. $\rho = a \csc \theta$.
18. $\rho = a \tan \theta$.

GROUP B

Draw the graph of each of the following equations.

19. $\rho = 3(2 + 3\cos\theta)$.

20. $\rho = 2(3 - 2\sin\theta)$.

21. $\rho = 3 - 2\sin\theta$.

22. $\rho^2 = 4\cos 2\theta$.

23. $\rho = 1 + 2\cos(\theta/2)$.

24. $\rho = 1 + \sin(\theta/2)$.

25. $\rho^2 = a^2\sin 2\theta$.

26. $\rho = a\sin 3\theta$.

27. $\rho = 8\sin^3(\theta/3)$.

28. $\rho^2 = a^2\sin 3\theta$.

29. $\rho = a(1 - \cos 2\theta)$

30. $\rho = a(1 + 2\cos 2\theta)$.

Transform each of the following equations to polar coordinates or to rectangular coordinates.

31. $x^2 + y^2 - 2ax + 2ay = 0$.

32. $(x^2 + y^2)^2 = a^2(x^2 - y^2)$.

33. $x^3 + y^3 - 3axy = 0$.

34. $\rho = a\sin 2\theta$.

35. $\rho^2 = a^2\cos 2\theta$.

36. $\rho = a\cos 2\theta$.

GROUP C

Draw the graph of each of the following equations.

37. $\rho = a\theta$.

38. $\rho = 8\cos^3(\theta/3)$.

39. $\rho^2 = a^2(\sin + \cos\theta)$.

40. $\rho\theta = a$.

41. $\rho = e^\theta$.

42. $\rho = 2\cos\theta - 3\sin\theta$.

Find the coordinates of the points of intersection of the following pairs of curves.

43. $\rho = 1 + \cos\theta$, $\quad \rho = 2\cos\theta$

44. $\rho^2 = a^2\sin\theta$, $\quad \rho^2 = a^2\sin 2\theta$.

45. $\rho^2 = a^2\sin\theta$, $\quad \rho^2 = a^2\sin 3\theta$.

Transform each of the following equations to polar coordinates.

46. $x = 2\cos\theta - \cos 2\theta$, $\quad y = 2\sin\theta - \sin 2\theta$.

47. $x = a\cos^3\theta$, $\quad y = a\sin^3\theta$

48. $x = \dfrac{\cos\theta}{\theta}$, $\quad y = \dfrac{\sin\theta}{\theta}$.

110. Angle between the Radius Vector and the Tangent.

Let the equation of the curve in Figure 87 be

$$\rho = f(\theta),$$

in which θ is expressed in radians. Let $P(\rho, \theta)$ be any point on the curve, where $OP = \rho$, and draw the tangent PT to the curve at this point. Finally, let the angle between the tangent and the radius vector be denoted by ψ, or

$$\angle OPT = \psi.$$

If the angle θ is given the increment $\Delta\theta$, the increment of the function
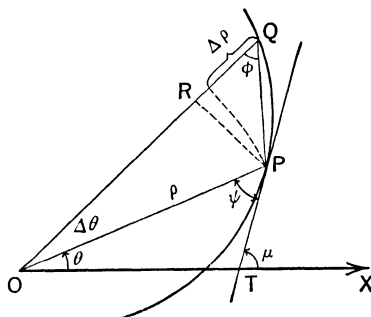
Fig. 87

is $\Delta\rho$, thus locating the point $Q(\rho + \Delta\rho, \ \theta + \Delta\theta)$ on the curve. The chord PQ is drawn and the line segment PR perpendicular to OQ through P is erected. The angle at Q in the right triangle PQR is represented by ϕ. Then, from this triangle,

$$\tan \phi = \frac{RP}{RQ} = \frac{RP}{OQ - OR} = \frac{\rho \sin \Delta\theta}{\rho + \Delta\rho - \rho \cos \Delta\theta}$$

$$= \frac{\rho \dfrac{\sin \Delta\theta}{\Delta\theta}}{\dfrac{\Delta\rho}{\Delta\theta} + \rho \dfrac{(1 - \cos \Delta\theta)}{\Delta\theta}}.$$

Since θ is expressed in radians,

$$\lim_{\Delta\theta \to 0} \frac{\sin \Delta\theta}{\Delta\theta} = 1 \quad \text{and} \quad \lim_{\Delta\theta \to 0} \frac{1 - \cos \Delta\theta}{\Delta\theta} = 0.$$

As $\Delta\theta$ approaches zero, the chord PQ approaches the tangent TP as a limiting position and angle ϕ approaches ψ. Hence,

$$\lim_{\Delta\theta \to 0} \tan \phi = \tan \psi$$

·and

$$\tan \psi = \frac{\rho}{\dfrac{d\rho}{d\theta}} = \rho \frac{d\theta}{d\rho}.$$

If the angle μ is the inclination of the tangent to the curve at the point P,

$$\mu = \theta + \psi.$$

From this relation the slope of the curve at the point P may be found by

$$\tan \mu = \frac{\tan \theta + \tan \psi}{1 - \tan \theta \tan \psi}.$$

To find the slope of the curve $\rho = 2(1 + \cos \theta)$ at the point where $\theta = \pi/6$, we first find $\tan \psi$.

$$\frac{d\rho}{d\theta} = -2 \sin \theta, \quad \tan \psi = \frac{1 + \cos \theta}{- \sin \theta}\bigg]^{\pi/6} = -(2 + \sqrt{3}\,).$$

Hence, $\qquad\qquad \tan \mu = -1 \quad \text{and} \quad \mu = \tfrac{3}{4}\pi.$

Exercise 78

GROUP A

1. Find the angle which the curve $\rho = a \cos^2 (\theta/2)$ makes with the polar axis.

2. Find the angle which the curve $\rho = 2(1 - 2 \sin \theta)$ makes with the polar axis

3. Find the angle which the curve $\rho = e^{a\theta}$ makes with the radius vector at any point.

4. Show that $\psi = \theta/3$ for any point on the curve $\rho = a \sin^3 (\theta/3)$.

Find the angle between each of the following curves and the given lines.

5. $\rho = 1 - 2 \cos 2\theta$ and $6\theta = \pi$.

6. $\rho = a \cos^2 \theta/2$ and $2\theta = \pi$.

7. $\rho(1 - \cos \theta) = 1$ and $4\theta = \pi$.

8. $\rho(1 + \cos \theta) = a$ and $2\theta = \pi$.

9. $\rho = a(1 - \cos \theta)$ and $2\theta = \pi$.

10. $\rho = a \cos 2\theta$ and $\theta = \arctan \frac{3}{4}$.

11. $\rho = a(1 - \cos \theta)$ and $\theta = \arctan \frac{3}{4}$.

GROUP B

Find the angle between each of the following pairs of curves.

12. $\rho^2 = a^2 \cos 2\theta$ and $\rho = a\sqrt{2} \sin \theta$.

13. $\rho^2 = a^2 \cos 2\theta$ and $\theta = 0$.

14. $\rho = a(1 + \sin \theta)$ and $\rho = a(1 - \sin \theta)$.

Find the slope of each of the following curves at the given points.

15. $\rho^2 = a^2 \cos 2\theta$ at $(a/\sqrt{2}, \pi/6)$.

16. $\rho = a(1 + \cos \theta)$ in the first quadrant for $\theta = \arcsin \frac{3}{5}$.

17. $\rho = a(1 + \sin \theta)$ in the first quadrant for $\theta = \arccos \frac{2}{5}$.

18. Find the polar equation of the tangent to the curve

$$\rho = 1 - \cos \theta \text{ at the point } (\tfrac{1}{2}, \pi/3).$$

111. Plane Area.

In Chapter VI the area under a curve in rectangular coordinates was expressed as the limit of a sum and by means of the fundamental theorem of integral calculus, the area was computed by a definite integral. In this section, application of the same theorem is made in finding plane areas bounded by curves expressed in polar coordinates. These continued applications of this theorem indicate, not only its importance in the calculus, but also the tremendous power of the definite integral.

Let the curve in Figure 88 be given by the equation

$$\rho = f(\theta),$$

where $f(\theta)$ is a continuous single-valued function from $\theta = \alpha$ to $\theta = \beta$ and is increasing with θ in this interval. We wish to find the area bounded by this curve and the lines $\theta = \alpha$ and $\theta = \beta$.

The angle $(\beta - \alpha)$ is divided into n equal angles $\Delta\theta$ by drawing radial lines through the pole O. These lines divide the area S into n increments ΔS. In the figure the increment ΔS_i is bounded by the lines OP_i, OP_{i+1} and the increment of arc length P_iP_{i+1}. If a circular arc P_iQ is drawn with O as a center and OP_i as a radius, the circular sector is said to be inscribed to the curve. The area of this sector differs from ΔS_i by infinitesimals of higher order than $\Delta\theta$. Hence, neglecting these infinitesimals, the element of area dS_i is the area of the circular sector. From elementary plane geometry,

$$dS = \tfrac{1}{2}\rho_i{}^2\,\Delta\theta.$$

Since

$$S = \sum_{i=1}^{i=n} \Delta S_i,$$

$$S = \lim_{n\to\infty}\sum_{i=1}^{i=n} dS_i = \lim_{\Delta\theta\to 0}\sum_{i=1}^{i=n}\tfrac{1}{2}\rho_i{}^2\,\Delta\theta.$$

By the fundamental theorem,

$$S = \frac{1}{2}\int_\alpha^\beta \rho^2\,d\theta,$$
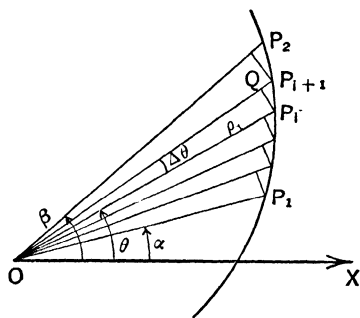
Fig. 88

where α, β and θ are expressed in radians.

The restriction made above, that the function increases with θ, was made to simplify the discussion. To illustrate that it is unnecessary, we shall find the area outside the circle $\rho = a$ and inside the circle $\rho = 2a\cos\theta$.

Let ρ' and ρ'' represent the radii vectores of points $P_i'(\rho_i',\theta_i)$ and $P_i''(\rho_i'',\theta_i)$ on the first and second curves, respectively.

Then the element of area is

$$dS = \tfrac{1}{2}(\rho_i''^2 - \rho_i'^2)\,\Delta\theta,$$

and the area is

$$S = \frac{1}{2}\lim_{\Delta\theta\to 0}\sum_{i=1}^{i=n}(4a^2\cos^2\theta_i - a^2)\,\Delta\theta.$$

The coordinates of the points of intersection of the circles are $(a, \pi/3)$ and $(a, -\pi/3)$. However, since both circles are symmetrical to the polar axis, it is convenient to find twice half the area. Now that the limits of the integral are known, the fundamental theorem is applied. Thus,

$$S = a^2\int_0^{\pi/3}(4\cos^2\theta - 1)\,d\theta = \frac{a^2}{6}(2\pi + 3\sqrt{3}).$$

Volume of a Solid of Revolution. To find the volume of the solid generated by rotating the area enclosed by the curve

$$\rho = 2a \sin \theta$$

about the polar axis, the cylindrical element (cylindrical shell) is chosen. Let the radius of the circular base and the length of the cylinder be y and $2x$ respectively. Then the element of volume is

$$dV = 4\pi xy\, dy.$$

Since

$$x = \rho \cos \theta = 2a \sin \theta \cos \theta,$$

$$y = \rho \sin \theta = 2a \sin^2 \theta$$

and

$$dy = 4a \sin \theta \cos \theta\, d\theta.$$

Then,

$$dV = 64\pi a^3 \sin^4 \theta \cos^2 \theta\, d\theta.$$

Hence

$$V = 64\pi a^3 \int_0^{\pi/2} \sin^4 \theta \cos^2 \theta\, d\theta$$

$$V = \tfrac{2}{3}\pi a^3 \Big[16 \sin^5 \theta \cos \theta - 4 \sin^3 \theta \cos \theta + 6\theta - 3 \sin 2\theta \Big]_0^{\pi/2}$$

$$= 2\pi^2 a^3.$$

Exercise 79

GROUP A

Find the entire area bounded by each of the following curves.

1. $\rho = 2a \cos \theta.$ **5.** $\rho^2 = a^2 \cos 2\theta.$

2. $\rho = a(1 + \cos \theta).$ **6.** $\rho = a \cos 2\theta.$

3. $\rho = 3 - \sin \theta.$ **7.** $\rho = a \cos 3\theta.$

4. $\rho = a(1 - \cos \theta).$ **8.** $\rho = a \sin 2\theta.$

9. Find the volume generated by rotating the upper half of the area inclosed by $\rho = 2a \cos \theta$ about the polar axis.

10. Find the area inside $\rho = 4 + 2 \cos \theta$ and outside $\rho = 4.$

GROUP B

Find the entire area bounded by each of the following curves.

11. $\rho = a \sin 3\theta.$ **14.** $\rho = a \sin^2 \theta.$

12. $\rho^2 = a^2 \cos \theta.$ **15.** $\rho^2 = 2 \sin \theta - 1.$

13. $\rho^2 = a^2 \sin 2\theta.$ **16.** $\rho^2 = a^2 \sin 3\theta.$

17. Find the area inside the smaller loop of the curve
$$\rho = 1 + 2 \sin \theta.$$

18. Find the area inside $\rho = a(1 + \cos \theta)$ and outside $\rho = 2a \cos \theta$.

19. Find the area of the segment of the parabola $\rho(1 + \cos \theta) = 2a$ from the vertex to the chord perpendicular to the axis.

20. Find the area common to $\rho = 3 \sin \theta$ and $\rho = 1 + \sin \theta$.

21. Find the volume generated by the rotation of the upper half of the area bounded by the curve $\rho = 1 + \cos \theta$ about the polar axis.

22. Find the angle of intersection of the curves $\rho = a \cos \theta$ and $\rho = a \sin 2\theta$.

23. Find the polar equation of the curve which intersects the radii vectores of its points at a constant angle.

24. Find the polar equation of the curve through the point $(3,0)$ if the radius vector and the curve at any point make an angle whose tangent is equal to the cube of the length of the radius vector.

112. Length of an Arc of a Curve.

From Section 87, the differential of the arc of a curve in rectangular coordinates is

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \sqrt{(dx)^2 + (dy)^2}.$$

This may be expressed in terms of polar coordinates by means of the transformation

$$x = \rho \cos \theta, \quad y = \rho \sin \theta.$$

Hence,

$$dx = \cos \theta\, d\rho - \rho \sin \theta\, d\theta$$

and

$$dy = \sin \theta\, d\rho + \rho \cos \theta\, d\theta.$$

Squaring and adding,

$$ds = \sqrt{\rho^2(d\theta)^2 + (d\rho)^2},$$

or

$$ds = \sqrt{\rho^2 + \left(\frac{d\rho}{d\theta}\right)^2}\, d\theta.$$

From this result, the length of the arc of a curve $\rho = f(\theta)$ between $\theta = \alpha$ and $\theta = \beta$ can be expressed by means of the definite integral

$$s = \int_C ds = \int_\alpha^\beta \sqrt{\rho^2 + \left(\frac{d\rho}{d\theta}\right)^2}\, d\theta,$$

where $f(\theta)$ is a continuous single-valued function in the interval.

Area of a Surface of Revolution. To find the area of the surface formed by revolving the lemniscate $\rho^2 = 4a^2 \cos 2\theta$, about the polar axis, we proceed as follows:

$$S = 2\pi \int_C y \, ds = 4\pi \int_0^{\pi/4} \rho \sin \theta \sqrt{\rho^2 + 4a^2 \frac{\sin^2 2\theta}{\cos 2\theta}} \, d\theta$$

$$= 16\pi a^2 \int_0^{\pi/4} \sin \theta \, d\theta = 8\pi a^2 (2 - \sqrt{2}).$$

Exercise 80

GROUP A

1. Find the total length of the curve $\rho = 2a \sin \theta$.
2. Find the total length of the curve $\rho = a(1 - \cos \theta)$.
3. Find the total length of the curve $\rho = a \cos^3 (\theta/3)$.
4. Find the length of the curve $\rho = a\theta$ from $\theta = 0$ to $\theta = 2\pi$.
5. Find the area of the surface generated by the rotation of the curve $\rho = 2a \cos \theta$ about the polar axis.
6. Find the area of the surface generated by the rotation of the curve $\rho^2 = 2a^2 \cos 2\theta$ about the polar axis.

GROUP B

7. Find the total length of the curve $\rho = \sin^3 (\theta/3)$.
8. Find the total length of the curve $\rho = a(1 + \sin \theta)$.
9. Find the length of the curve $\rho = e^{a\theta}$ from θ_1 to θ_2.
10. Find the area common to the two curves $\rho = 2a \cos \theta$ and $\rho = 2a \sin \theta$.
11. Find the area enclosed by the curve $\rho = a(1 + \cos \theta)$ which is cut off by the line through the point $(\tfrac{3}{4}a, 0)$ perpendicular to the polar axis.
12. Find the area of the surface generated by the rotation of the curve $\rho^2 = 2a^2 \cos 2\theta$ about the line $\theta = \pi/2$.

113. Plane Area by Double Integration.

Double integration was used in Section 93 in finding plane areas where the equations of the curves bounding those areas were expressed in rectangular coordinates. It is of interest to apply this same method of approach to the problem of finding plane areas when the curves involved are expressed in polar coordinates.

In Figure 89 the area S is bounded by the curves

$$\rho = f(\theta) \quad \text{and} \quad \rho = g(\theta)$$

and the lines $\theta = \alpha$ and $\theta = \beta$. Radial lines are drawn through O dividing the angle $(\beta - \alpha)$ into n equal angles $\Delta\theta$. Concentric circles are drawn with centers at O and with the radii of consecutive circles differing

by the equal length $\Delta\rho$. In this manner a set of elementary areas are formed of which $PQRS$ is one. Assuming the coordinates of the point P as $(\rho,\ \theta)$, the coordinates of the opposite vertex R are $(\rho + \Delta\rho,\ \theta + \Delta\theta)$.

The area of $PQRS$ is

$$\Delta S = \tfrac{1}{2}(\rho + \Delta\rho)^2\,\Delta\theta - \tfrac{1}{2}\rho^2\,\Delta\theta$$
$$= \rho\Delta\rho\,\Delta\theta + \tfrac{1}{2}\overline{\Delta\rho}^2\,\Delta\theta.$$

Neglecting the second infinitesimal, which is of higher order than is the first, the *element of area* is

$$dS = \rho\,\Delta\rho\,\Delta\theta.$$
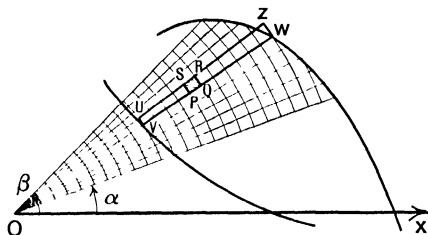
Fig. 89

The double sum of all such elements forms an arbitrarily close approximation to the area enclosed, by increasing the number of radial lines and the number of concentric circles. This may be shown as follows:

Referring to the same figure, let the radial line through P intersect the first curve at the point $V(\rho_i',\theta_i)$ and intersect the second curve at the
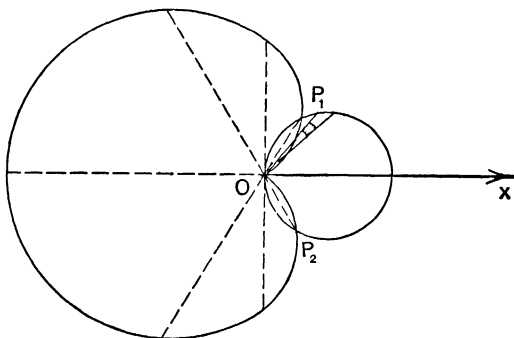
Fig. 90

point $W(\rho_i'',\theta_i)$. Holding θ_i and $\Delta\theta$ fixed, let $\Delta\rho$ approach zero. Thus the number m of concentric circles is increased without limit. The area of the strip $UVWZ$ is

$$\left[\lim_{\Delta\rho \to 0} \sum_{i=1}^{i=m} \rho_i\,\Delta\rho \right]\Delta\theta = \left[\int_{\rho'}^{\rho''} \rho\,d\rho \right]\Delta\theta.$$

As $\Delta\theta$ approaches zero, the number n of radial lines is increased without limit. Hence, the required area is

$$S = \lim_{\Delta\theta \to 0} \sum_{j=1}^{j=n} \left[\int_{f(\theta_j)}^{g(\theta_j)} \rho\,d\rho \right]\Delta\theta = \int_{\alpha}^{\beta} \int_{f(\theta)}^{g(\theta)} \rho\,d\rho\,d\theta.$$

In application, the area inside the circle $\rho = 2a \cos \theta$ and outside the cardioid $\rho = 2a(1 - \cos \theta)$ is found as follows:

The curves are drawn in Figure 90, where the symmetry with respect to the polar axis is shown and the points of intersection are indicated by $P_1(a,\pi/3)$ and $P_2(a, -\pi/3)$. The element of area is

$$dS = \rho \, \Delta\rho \, \Delta\theta.$$

Then
$$S = 2 \int_0^{\pi/3} \int_{2a\,(1-\cos\theta)}^{2a\cos\theta} \rho \, d\rho \, d\theta = \int_0^{\pi/3} \rho^2 \Big]_{2a(1-\cos\theta)}^{2a\cos\theta} d\theta$$

$$S = 4a^2 \int_0^{\pi/3} (2\cos\theta - 1) \, d\theta = \tfrac{1}{3}a^2(3\sqrt{3} - \pi).$$

Exercise 81

GROUP A

Find each of the following areas by double integration.

1. Bounded by $\rho = 6a \sin \theta$.
2. Bounded by $\rho^2 = a^2 \cos 2\theta$.
3. Bounded by $\rho = 1 - \sin \theta$.
4. Between the two curves $\rho = 3 \sin \theta$ and $\rho = 6 \sin \theta$.
5. Inside the curve $\rho = a \sin \theta$ and outside $\rho = 2a \cos \theta$.
6. Inside the curve $\rho = 2a \sin \theta$ and outside $\rho = a$.
7. The first quadrant area inside $\rho = a$ and outside $\rho^2 = a^2 \sin 2\theta$.

GROUP B

Find each of the following areas by double integration.

8. Inside the curve $\rho = a(1 + \cos \theta)$ and outside $\rho = 2a \cos \theta$.
9. The area common to the areas enclosed by the curves $\rho = 6a \sin \theta$ and $\rho = 2a(1 + \sin \theta)$.
10. Inside the curve $\rho = a$ and outside $\rho = a(1 - \cos \theta)$.
11. Bounded by the curves $\rho = 2a \sin \theta$ and $\rho^2 = 2a^2 \cos 2\theta$ and the lines $4\theta + \pi = 0$ and $4\theta - \pi = 0$.
12. Inside the curve $\rho^2 = 2a^2 \cos 2\theta$ and outside $\rho = a$.
13. Inside the curve $\rho = a(1 - \cos \theta)$ and outside $\rho(1 - \cos \theta) = a$.
14. Between the curves $\rho = a \cos 2\theta$ and $\rho = a(3 - \cos \theta)$ and the lines $6\theta + \pi = 0$ and $6\theta - \pi = 0$.

CHAPTER XIV

SOLID ANALYTIC GEOMETRY

114. Solid Analytic Geometry.

The study of the calculus as presented thus far has been largely concerned with functions of one variable. The content of plane analytic geometry, which deals with such functions, is used so consistently, that it is indispensable for adequate interpretations of much of the calculus. In precisely the same manner, the content of solid analytic geometry is essential for a proper understanding of much of the material presented in the remaining chapters of this book. Consequently, it is advisable that this chapter on analytic geometry of space be inserted for reference or for study.

Solid analytic geometry deals with lines, planes, curves and surfaces in a three dimensional space. In this chapter is to be found a very brief study of some of the equations of these loci.

115. Space Coordinate Systems.

There are three systems of reference, or *coordinate systems*, which are commonly used in a study of solid analytic geometry. They are the *rectangular*, the *cylindrical* and the *spherical* coordinate systems. The first is an extension of the rectangular Cartesian coordinate system in the plane and the latter is an extension of the polar coordinate system in the plane. The second system makes use of both the rectangular and the polar coordinate systems.

Rectangular Coordinates. A system of reference in space is established by means of three mutually perpendicular planes which intersect in three mutually perpendicular lines. These are called the *coordinate planes* and the *coordinate axes*, respectively, and the common point is the *origin*. The *rectangular coordinates* of a point are the three directed distances of that point from the three coordinate planes. Consequently, the position of a point is uniquely determined provided that the distances from the coordinate planes and the directions of those distances are known.

If x, y and z are the coordinates of a point $P(x,y,z)$, in Figure 91,

$$x = FP = OA, \quad y = EP = OB, \quad z = DP = OC.$$

Having directed the axes as indicated in the figure, any point whose first coordinate is positive is in front of the yz-plane, and any point whose first coordinate is negative is back of that plane. Similarly, the signs of the other two coordinates give positions to the right or to the left of the xz-plane and above or below the xy-plane. Thus a point may be located in any one of the eight *octants*. Any point whose second coordinate is zero lies on the xz-plane and one whose first two coordinates are zero lies on the z-axis. Thus every point in space has three coordinates which are positive or negative real numbers, including zeros.
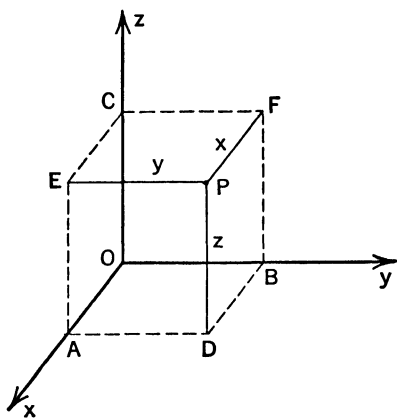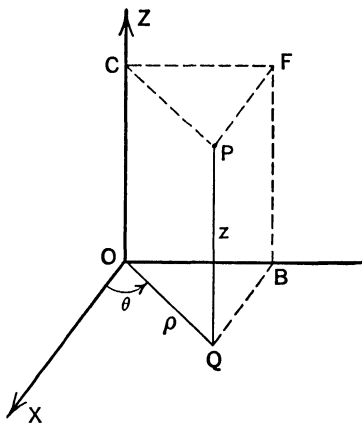
FIG. 91   FIG. 92

Cylindrical Coordinates. A point P whose projection on the horizontal plane is known by its polar coordinates $Q(\rho,\theta)$, is located in space if its directed distance z from that plane is also known. The coordinates of the point P in Figure 92 are (ρ,θ,z), where

$$\rho = OQ, \quad \theta = \angle XOQ, \quad z = QP = OC.$$

Such coordinate systems are called *cylindrical coordinates* and are particularly useful in expressing the locus of a point which lies on a surface of revolution about one of the coordinate axes.

While it is customary to establish a polar coordinate system in the horizontal plane and a vertical z-axis through the pole, this is but one of three possibilities since the polar coordinate system may be established in any one of the three mutually perpendicular planes.

It is often desirable to change the equation of a locus from the rectangular coordinate system to the cylindrical and vice versa. The relations which exist between the coordinates of the two systems depend on the relative position of the two sets of axes. If the x-axis is the polar axis and the z-axis is the vertical axis,

$$x = \rho \cos \theta, \quad y = \rho \sin \theta, \quad z = z$$

and
$$\rho^2 = x^2 + y^2, \quad \tan \theta = \frac{y}{x}.$$

Spherical Coordinates. Three mutually perpendicular lines OX, OY and OZ from a common point O may be used to establish a spherical coordinate system. In Figure 93 the *spherical coordinates* of point P are r, θ and ϕ, where

$$r = OP, \quad \theta = \angle XOD, \quad \phi = \angle ZOP.$$

The point P and the OZ-axis determine a plane which is located by the angle θ between that plane and the XZ-plane. The angle θ ranges from 0 to 2π for any position of P. In this plane, P is the point having the polar coordinates (r,ϕ) with O as the pole and the line OZ as the polar axis.

In comparing the cylindrical and spherical coordinates of a point, it is to be observed that ρ of the former is not equal to r of the latter, but that

$$\rho = r \sin \phi.$$

FIG. 93

As with cylindrical coordinates, it is often desirable to change the equation of a locus from the rectangular coordinate system to the spherical and vice versa. If the x-, y- and the z-axes are the lines OX, OY and OZ, respectively,

$$x = r \sin \phi \cos \theta, \quad y = r \sin \phi \sin \theta, \quad z = r \cos \phi.$$
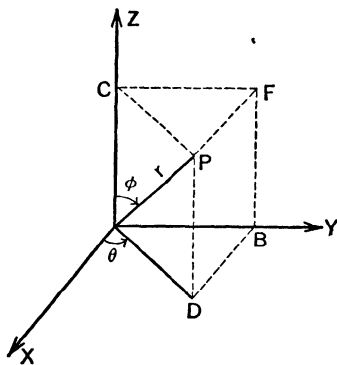
Exercise 82

GROUP A

Write the equations of the following loci in rectangular coordinates.

1. Each of the coordinate planes.
2. Each of the three planes parallel to the yz-plane, the xz-plane and the xy-plane and at distances a, b and c, respectively, from them.
3. Each of the two planes through the z-axis bisecting the angle between the xz- and the yz-planes.
4. Each of the two planes through the y-axis bisecting the angle between the xy- and the yz-planes.
5. Each of the two planes through the x-axis bisecting the angle between the xy- and the xz-planes.

Find the length of each of the following line segments.

6. Between $(1,2,2)$ and the origin.
7. Between $(1,2,1)$ and $(5,4,5)$.
8. Between $(3,2,-1)$ and $(-1,-3,2)$.
9. Between $(0,1,6)$ and $(2,0,-3)$.
10. Find the coordinates of the midpoint of the line segment joining $(3,2,-1)$ and $(7,8,5)$.

GROUP B

Write the equation of each of the following planes.

11. Parallel to the z-axis through the points $(a,0,0)$ and $(0,b,0)$.
12. Parallel to the y-axis through the points $(a,0,0)$ and $(0,0,c)$.
13. Parallel to the x-axis through the points $(0,b,0)$ and $(0,0,c)$.
14. Through the x-axis making an angle of $60°$ above the xy-plane.
15. Transform the equation $x^2 + y^2 + z^2 = a^2$ to cylindrical and to spherical coordinates
16. Transform the equation $\rho = a$ to rectangular and to spherical coordinates.
17. Transform the equations $z = \rho^2$, $\rho = 2a \sin \theta$ and $z = 2a - \rho \cos \theta$ to rectangular coordinates.
18. Transform the equation $r = a$ to rectangular and to cylindrical coordinates.
19. Transform the equation $z = ae^{x^2+y^2}$ to cylindrical coordinates.
20. Find the perpendicular distances of the point $P(\rho,\theta,z)$ from the axes OX, OY and OZ.

116. Direction Cosines.

A directed line through the origin of a rectangular coordinate system makes three angles α, β and γ with the x-axis, the y-axis and the z-axis, respectively. These angles are called the *direction angles* and their cosines are called the *direction cosines* of the line. Any line which is parallel to such a line and which has the same direction, has the same direction angles and the same direction cosines.

The sum of the squares of the direction cosines of any line is equal to unity.

In Figure 94 the line AB is directed in such a way that the direction angles are acute angles making the direction cosines positive. However, were it oppositely directed, each angle would be obtuse and the direction cosines would be negative. The line OP of any length r is drawn through the origin parallel to AB and having the same direction. The direction angles of OP, which are the same as those of AB, are α, β and γ, indicated in the figure. If the coordinates of P are taken to be (x,y,z) then
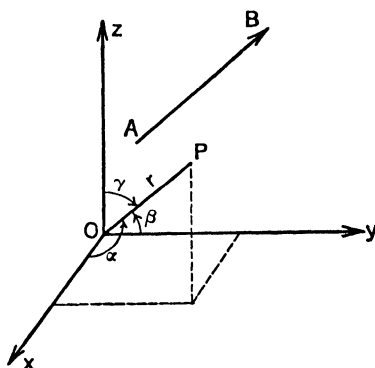
Fig. 94

$$x = r \cos \alpha, \quad y = r \cos \beta, \quad z = r \cos \gamma.$$

But, since $\qquad x^2 + y^2 + z^2 = r^2$,

$$r^2 \cos^2 \alpha + r^2 \cos^2 \beta + r^2 \cos^2 \gamma = r^2$$

and

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1.$$

Any three numbers which are proportional to the direction cosines of a line are known as *direction numbers* of that line. Let a, b and c represent three such numbers, then

$$\frac{\cos \alpha}{a} = \frac{\cos \beta}{b} = \frac{\cos \gamma}{c} = k,$$

or

$$\cos \alpha = ak, \quad \cos \beta = bk, \quad \cos \gamma = ck.$$

Substituting these values of the cosines in the relation above, we have

$$a^2 k^2 + b^2 k^2 + c^2 k^2 = 1.$$

Hence, the proportionality factor is

$$k = \frac{1}{\pm \sqrt{a^2 + b^2 + c^2}}$$

and

$$\cos \alpha = \frac{a}{\pm \sqrt{a^2 + b^2 + c^2}}, \quad \cos \beta = \frac{b}{\pm \sqrt{a^2 + b^2 + c^2}},$$

$$\cos \gamma = \frac{c}{\pm \sqrt{a^2 + b^2 + c^2}}.$$

Since the direction angles are fixed for any directed line, a proper choice of sign must be made for each direction cosine.

The direction cosines of a line segment are proportional to the lengths of the projections of the line segment on the coordinate axes.
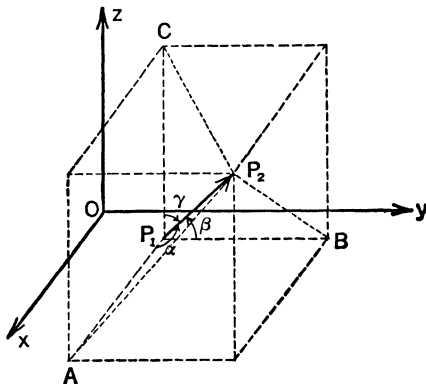
FIG. 95

In Figure 95 a rectangular parallelopiped is constructed, having the given directed line segment P_1P_2 as a diagonal, with its edges parallel to the coordinate axes. If the coordinates of P_1 and P_2 are (x_1,y_1,z_1) and (x_2,y_2,z_2), respectively,

$$P_1A = x_2 - x_1, \quad P_1B = y_2 - y_1, \quad P_1C = z_2 - z_1.$$

From the right triangles P_1AP_2, P_1BP_2 and P_1CP_2,

$$\cos \alpha = \frac{P_1A}{P_1P_2}, \quad \cos \beta = \frac{P_1B}{P_1P_2}, \quad \cos \gamma = \frac{P_1C}{P_1P_2}.$$

Hence, direction numbers of the given line segment are

$$a = x_2 - x_1, \quad b = y_2 - y_1, \quad c = z_2 - z_1.$$

The length of the line segment P_1P_2 is

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

117. Angle between Two Lines.

The *angle* between two intersecting directed lines in space is defined as the angle between the positive directions of those lines.

Let CA and CB be two intersecting lines in space, directed as indicated in Figure 96. If two lines are drawn through the origin parallel to the given

lines and a point is chosen on each,

$$\angle BCA = \angle P_2 O P_1 = \mu.$$

The coordinates of P_1 and P_2 being (x_1,y_1,z_1) and (x_2,y_2,z_2), respectively,

$$\overline{OP_1}^2 = x_1^2 + y_1^2 + z_1^2, \quad \overline{OP_2}^2 = x_2^2 + y_2^2 + z_2^2$$

and

$$\overline{P_1P_2}^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2,$$
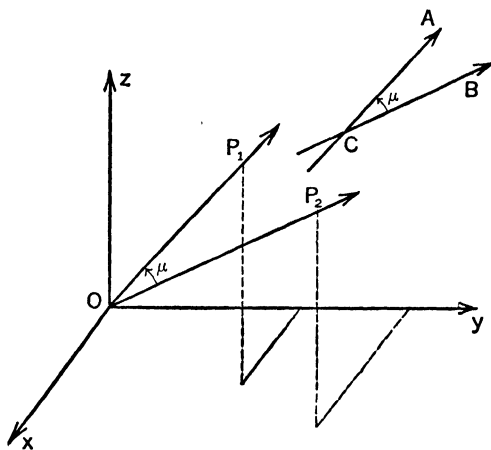
FIG. 96

From the law of cosines

$$\overline{P_1P_2}^2 = \overline{OP_1}^2 + \overline{OP_2}^2 - 2\overline{OP_1} \cdot \overline{OP_2} \cos \mu.$$

Hence,

$$\cos \mu = \frac{\overline{OP_1}^2 + \overline{OP_2}^2 - \overline{P_1P_2}^2}{2OP_1 \cdot OP_2},$$

or

$$\cos \mu = \frac{x_1 x_2 + y_1 y_2 + z_1 z_2}{\sqrt{x_1^2 + y_1^2 + z_1^2} \sqrt{x_2^2 + y_2^2 + z_2^2}},$$

and

$$\cos \mu = \cos \alpha_1 \cos \alpha_2 + \cos \beta_1 \cos \beta_2 + \cos \gamma_1 \cos \gamma_2.$$

If two intersecting lines in space are perpendicular,

$$\cos \mu = 0$$

and

$$x_1 x_2 + y_1 y_2 + z_1 z_2 = 0.$$

Hence, *the sum of the products of corresponding direction numbers of two perpendicular lines is zero.*

If we have given the quadrilateral $A(3,3,4)$, $B(2,4,6)$, $C(-2,4,7)$ and $D(0,2,3)$, the sides AB and CD are shown to be parallel and the angles A and D to be right angles as follows:

Direction numbers of the sides are

$$AB : -1,1,2. \quad CB : 4,0,-1. \quad DC : -2,2,4. \quad DA : 3,1,1.$$

Since the direction numbers of AB and DC are proportional, the lines are parallel and the four points are coplanar. The other two lines are not parallel and the figure is a trapezoid. Taking the sum of the products of the direction numbers of AB and DA and of the direction numbers of DA and DC,

$$-3 + 1 + 2 = 0, \quad -6 + 2 + 4 = 0.$$

Hence the angles A and D are right angles.

Exercise 83

GROUP A

1. Find the length of each of the following line segments: $(3,1,-2)$ and $(1,-2,0)$. $(1,0,-2)$ and $(3,-1,-3)$. $(3,2,7)$ and $(0,-2,7)$

2. Find direction numbers of each of the line segments given in Problem 1.

3. The line segment $P_1(1,2,3)$ $P_2(2,-1,2)$ is directed from P_1 to P_2. Find direction numbers and the direction cosines of the line segment.

4. Show that the three points $(1,-2,3),(-2,4,-6)$ and $(6,-12,18)$ are collinear.

5. Is it possible for a line through the origin to make the angles $120°$, $120°$ and $60°$ with the coordinate axes?

6. If $\alpha = 45°$ and $\beta = \gamma$, find the angle β.

7. The line segment $OP = 5$ and $\alpha = \beta = 60°$. Find the coordinates of P and $\cos \gamma$.

8. Show that the points $(0,0,0)$, $(2,1,4)$, and $(5,0,6)$ and $(3,-1,2)$ lie in a plane and form a parallelogram.

GROUP B

9. Find the angles of the triangle $(4,3,1)$, $(2,6,-5)$, $(-1,0,-7)$.

10. Find the area of the triangle given in Problem 9.

11. Find the direction angles of a line making equal acute angles with the coordinate axes

12. Find the coordinates of the midpoint of the line segment $P_1(x_1,y_1,z_1)$ $P_2(x_2,y_2,z_2)$.

13. Show that the coordinates of a point P_0 which divides the line segment P_1P_2 in the ratio r_1/r_2 are

$$\left[\frac{r_2x_1 + r_1x_2}{r_1 + r_2}, \frac{r_2y_1 + r_1y_2}{r_1 + r_2}, \frac{r_2z_1 + r_1z_2}{r_1 + r_2} \right].$$

14. Show that the points $(0,0,0)$, $(1,1,-2)$, $(-1,1,-3)$ are the vertices of a right triangle.

15. Find direction numbers of a line perpendicular to the plane of the triangle given in Problem 14.

16. Prove that direction numbers of a line perpendicular to each of two non-parallel lines having direction numbers a_1,b_1,c_1, and a_2,b_2,c_2 are given by

$$\left| \begin{array}{cc} b_1 & c_1 \\ b_2 & c_2 \end{array} \right|, \quad \left| \begin{array}{cc} c_1 & a_1 \\ c_2 & a_2 \end{array} \right|, \quad \left| \begin{array}{cc} a_1 & b_1 \\ a_2 & b_2 \end{array} \right|.$$

118. Equations of Planes.

A plane is determined in space by three conditions, such as being required to pass through three points, or to pass through one point and be perpendicular to a given line. These geometric conditions can be expressed in the form of an equation containing the constants which locate the plane in space and the coordinates of the moving point which describes the plane. This equation is called the *equation of the plane*. It will be found that all forms of the equation of a plane have one characteristic in common, that of being a first degree equation in any one, or any two, or all three variables x, y and z.
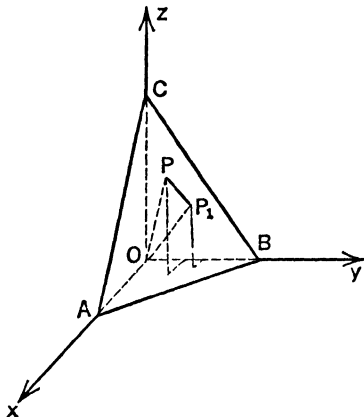
Fig. 97

In Figure 97 let the point $P(x,y,z)$ be any point of the plane ABC and let OP_1 be the perpendicular to the plane from the origin, where the point $P_1(x_1,y_1,z_1)$ also lies on the plane. If the perpendicular OP_1 has the length p and the direction angles α, β and γ, the coordinates of P_1 are

$$x_1 = p \cos \alpha, \quad y_1 = p \cos \beta, \quad z_1 = p \cos \gamma.$$

Then direction numbers of the line P_1P are

$$x - p \cos \alpha, \quad y - p \cos \beta, \quad z - p \cos \gamma.$$

But since the lines OP_1 and P_1P are perpendicular for all positions of the point P,

$$\cos \alpha \, (x - p \cos \alpha) + \cos \beta \, (y - p \cos \beta) + \cos \gamma \, (z - p \cos \gamma) = 0,$$

or

$$x \cos \alpha + y \cos \beta + z \cos \gamma = p.$$

This equation is satisfied by the coordinates of all points of the plane and by those points only. Hence, *the equation of a plane is always of the first degree in x, y and z.*

The form of the equation of a plane which is derived above is known as the *normal form* of the equation. Any equation of a plane may be written in this form by dividing it by the square root of the sum of the squares of the coefficients of the variables. Since p is always taken to be positive, the sign of the radical is chosen so that the constant term of the equation is positive in the right-hand member. For example, the equation

$$2x - 2y + z + 9 = 0$$

is written in the normal form by dividing by -3:

$$-\tfrac{2}{3}x + \tfrac{2}{3}y - \tfrac{1}{3}z = 3.$$

The direction cosines of any normal to the given plane are the coefficients of this equation and its distance from the origin is the constant term.

The *coefficients A, B and C of the general form of the equation of a plane,*

$$Ax + By + Cz + D = 0,$$

are direction numbers of a normal to the plane.

The *traces of a plane* are the lines of intersection formed by it and each of the coordinate planes. The equations of these lines are found by letting x, y and z, each equal to zero in turn in the equation of the plane. For example, the xz-trace of a plane is

$$Ax + Cz + D = 0, \quad y = 0.$$

The *intercepts of a plane* are the directed lengths cut off on each of the coordinate axes. An intercept is found by letting two of the variables of the equation equal zero simultaneously.

The equation of a plane in which one variable is missing expresses the fact that one intercept is missing. Such a plane is parallel to that axis. For example, the equation of a plane written,

$$By + Cz + D = 0,$$

represents a plane parallel to the x-axis. Similarly, the equation of a plane in which two variables are missing expresses the fact that two intercepts are missing. Such a plane is parallel to a coordinate plane.

Exercise 84

GROUP A

1. Write each of the following equations in the normal form. Find the distance of each plane from the origin and the direction cosines of its normal.
$$x + 2y - 2z = 9, \ 2x - y - 2z + 12 = 0, \ 3x - 4y + 10 = 0.$$

2. Find the distance of each of the planes given in Problem 1 to the point $(1,1,-1)$.

3. Find the equation of the plane through the point $P(2,1,2)$ and perpendicular to the line OP.

4. Find the three traces and the three intercepts of each of the following planes.
$$2x - 3y + z = 6, \ 3x + 4y - 2z = 8 \text{ and } 5x + 3y + z + 10 = 0.$$

5. Show that the planes $2x - y + 3z = 4$ and $6x - 3y + 9z + 3 = 0$ are parallel.

6. Show that the planes $2x - y + 3z = 5$ and $3x + 3y - z = 8$ are perpendicular.

Find the equation of each of the following planes.

7. Through the points $(2,0,0)$, $(0,1,0)$, $(0,0,3)$.

8. Through the points $(1,1,1)$, $(-1,2,-3)$, $(2,-1,0)$.

9. Through the point $(2,1,3)$ perpendicular to a line having direction numbers $2, 2, -1$.

10. Distance from the origin ± 2 and $\cos \alpha = \frac{4}{9}$, $\cos \gamma = -\frac{8}{9}$.

GROUP B

11. Derive the *intercept form* of the equation of a plane $x/a + y/b + z/c = 1$ by writing the equation of a plane through the points $(a,0,0)$, $(0,b,0)$, $(0,0,c)$.

12. Find the coordinates of the point of intersection of the planes $x + y - z = 6$, $2x - 4y + z = 4$, $3x + y + 2z = 2$.

13. Find the equation of the plane having intercepts $a = 4$, $b = -3$ and $c = -2$.

14. Find the equation of the plane through the point $(3,-2,1)$ parallel to the plane $2x - 2y + z = 4$.

15. Find the equation of the plane through the points $(2,1,3)$, $(1,2,1)$ perpendicular to the plane $x + 2y - 2z = 4$

16. Find the cosine of the angle between the planes
$$2x - 2y + z = 3 \quad \text{and} \quad x + 2y - 2z = 8.$$

17. Show that the plane $x + y + z = 12$ intersects the coordinate planes at equal angles and that the sine of those angles is $\sqrt{\frac{2}{3}}$.

18. Show that the distance from the plane $x \cos \alpha + y \cos \beta + z \cos \gamma = p$ to the point (x_1,y_1,z_1) is equal to $x_1 \cos \alpha + y_1 \cos \beta + z_1 \cos \gamma - p$.

19. Find the distance from the plane $x + 2y - 2z = 12$ to each of the points $(1,2,1)$, $(1,0,2)$, $(0,0,3)$.

20. Find the equations of the locus of a point equidistant from the planes
$$2x - 2y + z = 3 \quad \text{and} \quad x - 2y - 2z = 7.$$

119. Equations of Lines.

A line in space is located by any pair of planes through it. The equations of two such planes are known as the equations of the line of intersec-

tion. Thus, the general form of the equations of a line may be written

$$\begin{cases} A_1x + B_1y + C_1z + D_1 = 0 \\ A_2x + B_2y + C_2z + D_2 = 0. \end{cases}$$

A line in space is determined by one of its points and its direction angles. Let $P_1(x_1,y_1,z_1)$ be a given point on a line and $P(x,y,z)$ any other point. If $P_1P = d$ and α, β and γ are the direction angles of the line,

$$d \cos \alpha = x - x_1, \quad d \cos \beta = y - y_1, \quad d \cos \gamma = z - z_1.$$

Equating the values of d from these equations,

$$\frac{x - x_1}{\cos \alpha} = \frac{y - y_1}{\cos \beta} = \frac{z - z_1}{\cos \gamma}.$$

If direction numbers of the line are used, a more convenient form of the equations of the line is obtained:

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}.$$

This form of the equation of a line which has been derived is known as the *symmetric form*, which represents the line by means of three planes, one parallel to each coordinate axis.

Let us reduce the equations of a line

$$\begin{cases} 4x + y - z + 2 = 0 \\ x + 4y + 2z - 1 = 0 \end{cases}$$

to the symmetric form and hence find its direction numbers. Solving the two equations simultaneously by eliminating first y and then z,

$$5x = 2z - 3, \quad -3x = 2y + 1.$$

From these equations we have

$$\frac{x - 0}{2} = \frac{y + \frac{1}{2}}{-3} = \frac{z - \frac{3}{2}}{5}.$$

Exercise 85

1. Find the direction cosines and the direction angles of the line

$$x + 3 = y - 2 = \frac{z - 1}{\sqrt{2}}.$$

2. Find the direction cosines of the line $\dfrac{x - 2}{2} = \dfrac{y - 3}{2} = z + 3$ and the coordinates of the points of intersection with the coordinate planes.

3. Write the equations of the line $\begin{cases} 2x - y - 2z + 2 = 0 \\ x + y - 4z + 3 = 0 \end{cases}$, in the symmetric form and find its direction cosines.

4. Find the angle between the lines

$$\frac{x-1}{2} = \frac{y+2}{3} = \frac{z-3}{6} \quad \text{and} \quad \frac{x-1}{6} = \frac{y+2}{9} = \frac{z-3}{2}.$$

5. Show that the line $\dfrac{x-2}{3} = \dfrac{y-2}{2} = \dfrac{z+1}{1}$ lies in plane M_1, is parallel to plane M_2 and is perpendicular to plane M_3. $\quad M_1 : 2x - y - 4z = 6.'$

$$M_2 : 4x - 2y - 8z = 15. \quad M_3 \cdot 3x + 2y + z = 17.$$

Find the equations of each of the following lines.

6. Through the points $(2, -3, 4)$ and $(1, 2, 3)$.

7. Through the point $(4, -3, 2)$ parallel to the line in Problem 6.

8. Through the point $(1, -2, 3)$ perpendicular to the plane $3x - 5y - 6z = 4$.

9. Through the point $(-2, 3, -4)$ having $\cos \beta = \frac{1}{2}$ and $\cos \gamma = \frac{1}{4}$.

10. Through the point $(-1, 1, 2)$: (a) parallel to the x-axis, (b) parallel to the z-axis, (c) perpendicular to the x-axis, (d) perpendicular to the y-axis.

11. Find the equations of the projections on the yz-, the xz- and the xy-planes of the lines of intersection of the plane $2x + 3y + 4z = 12$ with the planes $x - 3 = 0$, $y - 2 = 0$ and $z - 1 = 0$, respectively

12. Show that the lines of intersection of $3x - 4y + 2z = 12$ with the planes $x = a$, $y = b$ and $z = c$, each form a series of parallel lines. Find a, b and c so that the three projections of the lines pass through the origin.

120. Surfaces and Curves.

An equation in three variables may be represented by either of the symbols

$$F(x, y, z) = 0, \quad z = f(x, y).$$

The locus of all points whose coordinates satisfy such an equation is known as a *surface*. The plane is the simplest case of a surface where the equation is linear in the three variables.

Points of a surface may be located by assigning particular values to x and y and computing the corresponding value or values of z. For the equation $z = f(x, y)$, such values of z will be distinct, finite in number and will lie on a line parallel to the z-axis. As other values are assigned to x and y, new lines parallel to the z-axis are located on which there are isolated points of the locus, in general. Hence, the locus has extension in two dimensions only, that is, it has no thickness. It is for this reason that the locus of an equation in three variables is known as a surface.

As with lines in space, curves in space cannot be represented by means of a single equation. The analytic equivalent for such a curve is a pair of

equations of two surfaces which intersect in the curve. The equations of a curved surface and a plane, considered simultaneously, are the equations of a *plane curve*. The equations of two curved surfaces, considered simultaneously, are the equations of a *space curve*.

The *traces of a surface* are the curves of intersection formed by it with the three coordinate planes. Other *plane sections* of surfaces are the curves obtained by taking intersecting planes parallel to the coordinate axes. The equations of plane sections are found from the equation of the surface by letting x, y and z equal to certain constants separately. The use of such sections is extremely helpful to obtain complete knowledge of the surface.

121. Right Cylinders.

A surface generated by a straight line which moves parallel to a fixed line and which intersects a given plane curve is a *cylinder*. The fixed line is known as the *directrix* and the curve, as the *generatrix*.

If the z-axis is the directrix of a cylinder and the circle $x^2 + y^2 = a^2$, $z = 0$ is its generatrix, we have a right circular cylinder whose equation is

$$x^2 + y^2 = a^2.$$

Again, if the x-axis is the directrix of a cylinder and the parabola $y^2 = 4pz$, $x = 0$ is its generatrix, we have a right parabolic cylinder whose equation is

$$y^2 = 4pz.$$

In solid analytic geometry, an equation of higher degree than the first containing two variables only, is the equation of a cylinder whose directrix is the axis of the missing variable and whose generatrix is the plane curve having the same equation as the cylinder.

122. The Sphere.

A *sphere* is defined as the locus of a point in space whose distance from a fixed point is constant. The fixed point is the *center* of the sphere and the constant distance is its *radius*.

If the center of a sphere is $C(h,k,l)$ and the radius a, the equation which must be satisfied by the coordinates of any point $P(x,y,z)$ is

$$(x - h)^2 + (y - k)^2 + (z - l)^2 = a^2.$$

If the center of the sphere is at the origin, the equation of the sphere reduces

to
$$x^2 + y^2 + z^2 = a^2,$$
$$\rho^2 + z^2 = a^2$$
and
$$r = a$$

in rectangular, cylindrical and spherical coordinates, respectively.

123. Surfaces of Revolution.

A *surface of revolution* is one which is generated by the rotation of a plane curve about a straight line in its plane called the *axis* of the surface. The plane sections of a surface of revolution perpendicular to the axis are the circles known as *parallels*.

FIG 98

A parabola which is rotated about its axis generates a surface which is called a *paraboloid of revolution*. The equation of such a paraboloid may be derived as follows:

Consider the equation of a parabola in the *yz*-plane,

$$z^2 = 4ay, \quad x = 0,$$

and let the point $P(x,y,z)$ be any point on the surface generated by rotating the curve about the *y*-axis. In Figure 98 the plane *SBQ* is passed through the point P, parallel to the *xz*-plane, intersecting the given parabola in the point $Q(0,y,q)$. Since Q is a point of the parabola, its coordinates satisfy the given equation,

$$q^2 = 4ay.$$

From the right triangle PRB, $x^2 + z^2 = q^2$. Substituting the value of q^2, we have

$$x^2 + z^2 = 4ay.$$

An ellipse which is rotated about one of its axes generates a surface which is called a *spheroid*. If the major axis is the axis of rotation, the spheroid is known as *prolate*. If the minor axis is the axis of rotation, the spheroid is known as *oblate*. The equation of a spheroid may be derived as follows:

Consider the equation of an ellipse in the *xz*-plane,

$$\frac{x^2}{a^2} + \frac{z^2}{c^2} = 1, \quad y = 0,$$

and let the point $P(x,y,z)$ be any point on the surface generated by rotating the curve about the *z*-axis. In Figure 99 the plane *STQ* is passed through the point P, parallel to the *xy*-plane, intersecting the given ellipse in the
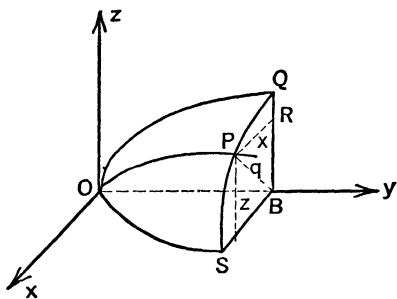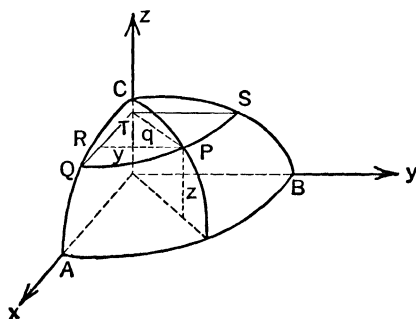
FIG. 99

point $Q(q,0,z)$. Hence, the coordinates of Q must satisfy the given equation,

$$\frac{q^2}{a^2} + \frac{z^2}{c^2} = 1.$$

From the right triangle PRT, $x^2 + y^2 = q^2$. Substituting,

$$\frac{x^2 + y^2}{a^2} + \frac{z^2}{c^2} = 1.$$

A hyperbola which is rotated about one of its axes generates a surface which is called a *hyperboloid*. If the transverse axis is the axis of rotation, the hyperboloid has two *nappes*. If the conjugate axis is the axis of rotation, the hyperboloid has one *nappe*. The equations of these surfaces are given in Section 127.

Exercise 86

1. Derive the equation of a sphere in rectangular coordinates having its center at (h,k,l) and radius a.
2. Derive the equations of a sphere of radius a with its center at the origin in cylindrical coordinates and in spherical coordinates.
3. Identify the following surfaces: $4x^2 + 9y^2 = 36$, $z^2 = 8y$, $9x^2 - 4z^2 = 36$, $y^2 + z^2 - 2y - 2z - 7 = 0$, $x^2 - 2x - 4z + 8 = 0$. Locate the surfaces relative to the coordinate axes.
4. Identify the following surfaces: $\rho = a$, $\rho = 2a \cos\theta$, $\rho = a(1 - \cos\theta)$, $\rho\cos\theta = a$, $\rho = 4z$. Locate the surfaces relative to the z-axis and the $\rho\theta$-plane.

Find the equations of each of the following surfaces of revolution.

5. The ellipse $b^2x^2 + a^2y^2 = a^2b^2$, $z = 0$, rotated about the x-axis.
6. The ellipse in Problem 5 rotated about the y-axis.
7. The parabola $z^2 = 4y$, $x = 0$, rotated about the z-axis.
8. The hyperbola $c^2x^2 - a^2z^2 = a^2c^2$, $y = 0$, rotated about the x-axis.
9. The hyperbola in Problem 8 rotated about the z-axis.
10. The line $bx + ay = ab$, $z = 0$, rotated about the y-axis.

GROUP B

11. Find the equation of the locus of a point (x,y,z) so that the sum of its distances from the points $(c,0,0)$ and $(-c,0,0)$ is equal to $2a$.
12. Find the equation of the locus of a point (x,y,z) so that the difference of its distances from the points $(0,c,0)$ and $(0,-c,0)$ is equal to $2a$.
13. Find the equation of the locus of a point (x,y,z) so that its distances from the plane $z + p = 0$ is equal to its distance from the point $(0,0,p)$.

Find the equation of each of the following surfaces of revolution.

14. $y^2 = 4x$, $z = 0$, rotated about $x + 2 = 0$, $y = z = 0$.
15. $2y - 3z = 6$, $x = 0$, rotated about $y = 5$, $x = z = 0$.
16. $3x^2 - 2z = 1$, $y = 0$, rotated about $x = 1$, $y = z = 0$.

17. Show that the sections of the surface $x^2 + y^2 = 8z$ parallel to the xy-plane are circles and that the sections parallel to the xz- and yz-planes are parabolas. Find the equation of the xy-projection of the circle having the focus of the paraboloid of revolution as center. Find the coordinates of the vertex and the focus of the yz-projection of the section made by the plane $x - 4 = 0$.

18. Draw a figure for the paraboloid given in Problem 17 and find the volume within the surface and the plane $z = 8$.

19. Show that the sections of the surface $4x^2 + 9y^2 + 4z^2 = 36$ parallel to the xz-plane are circles and that the sections parallel to the xy- and yz-planes are ellipses. Find the equation of the xz-projection of the circle having a focus of the ellipsoid of revolution as center. Find the coordinates of the foci of the xy-projection of the section made by the plane $z - 1 = 0$.

20. Draw a figure for the ellipsoid given in Problem 19 and find its volume.

124.   The Right Circular Cone.

A surface generated by a straight line which passes through a fixed point and which intersects a given plane curve is a *cone*. The fixed point is the *vertex* of the cone and the curve is its *generatrix*. If the generatrix is a circle and the vertex lies on a line through the center of the circle and perpendicular to its plane, the cone is a *right circular cone*.
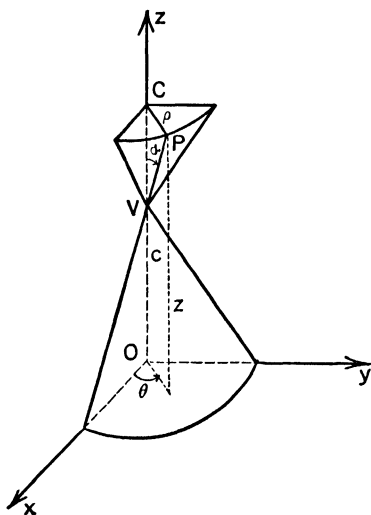
FIG. 100

The right circular cone in Figure 100 is taken with its vertex at the point $V(0,0,c)$ and with the generatrix a circle in the $\rho\theta$-plane whose center is at 0, so that the generating line makes a constant angle α with the Z-axis. To derive the equation of such a cone, any point $P(\rho,\theta,z)$ in cylindrical coordinates is taken on the surface. From the right triangle VCP,

$$\rho = (z - c) \tan \alpha.$$

If the vertex of the cone is at the origin, its equation reduces to

$$\rho = z \tan \alpha.$$

These two equations in rectangular and in spherical coordinates are as follows:

$$x^2 + y^2 = (z - c)^2 \tan^2 \alpha, \quad x^2 + y^2 = z^2 \tan^2 \alpha$$

and

$$(r \cos \phi - c) \tan \alpha = r \sin \phi, \quad \phi = \alpha,$$

respectively.

125. Quadric Surfaces.

A surface whose equation in the three variables x, y and z is of the second degree is called a *quadric surface*. The quadrics hold the same relative position of importance among surfaces that the conics have among curves of the plane. The cylinders, the cone, the sphere and the surfaces of revolution considered previously, are quadric surfaces.

There are nine types of quadric surfaces which are as follows:

The *ellipsoid* of which the sphere and the prolate and oblate spheroids are special cases.

The *hyperboloids* of which there are two species, one nappe and two nappes.

The *paraboloids* of which there are two species, the *elliptic* and the *hyperbolic*.

The *quadric cylinders* of which there are three species, the elliptic, the parabolic and the hyperbolic. The circular cylinder is a special case of the elliptic cylinder.

The *quadric cone* of which the circular cone is a special case.

126. The Ellipsoid.

The surface whose equation in rectangular coordinates is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

is called an *ellipsoid*. If two of the constants are equal, the ellipsoid is a spheroid, and if the three are equal, it is a sphere.

To derive the equation of the ellipsoid, we may proceed as follows:

In Figure 101 the two ellipses AQB and ASC are taken in the xy- and the xz-planes, respectively. They have the one axis OA in common. The equations of the ellipses may be assumed to be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad z = 0$$

and

$$\frac{x^2}{a^2} + \frac{z^2}{c^2} = 1, \quad y = 0.$$
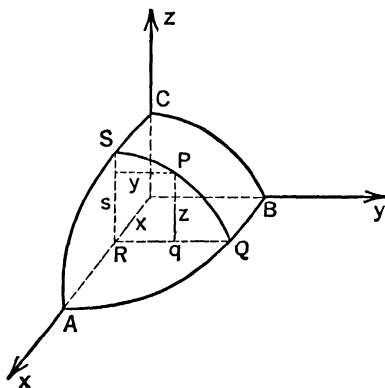
Fig. 101

If a variable ellipse through a point $P(x,y,z)$ is allowed to move so that the

ends of its axes $Q(x,q,0)$ and $S(x,0,s)$ are on the fixed ellipses and if it lies in a plane through P always perpendicular to the x-axis, this ellipse generates the ellipsoid.

Since the points Q and S lie on the first and the second ellipses, respectively, their coordinates must satisfy the above equations,

$$\frac{x^2}{a^2} + \frac{q^2}{b^2} = 1, \quad q^2 = \frac{b^2}{a^2}(a^2 - x^2)$$

and

$$\frac{x^2}{a^2} + \frac{s^2}{c^2} = 1, \quad s^2 = \frac{c^2}{a^2}(a^2 - x^2).$$

The equation of the ellipse QPS, referred to the lines RQ and RS as the y- and the z-axes, respectively, is

$$\frac{y^2}{q^2} + \frac{z^2}{s^2} = 1.$$

Substituting the values obtained for q^2 and s^2, we have the equation of the surface,

$$b^2c^2x^2 + a^2c^2y^2 + a^2b^2z^2 = a^2b^2c^2.$$

127. Hyperboloids.

The surface whose equation in rectangular coordinates is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1,$$

is called the *hyperboloid of one nappe.* Such a quadric surface is represented in Figure 102 .

Fig. 102

The derivation of this equation may be carried out in a manner similar to the method used in the derivation of the equation of the ellipsoid. The two fixed hyperbolas in the xz- and yz-planes have a common axis and their equations may be written

$$c^2x^2 - a^2z^2 = a^2c^2, \quad y = 0, \quad \text{and} \quad c^2y^2 - b^2z^2 = b^2c^2, \quad x = 0.$$

The variable ellipse QPS is allowed to vary so that the ends of its axes, $Q(0,q,z)$ and $S(s,0,z)$, move on the hyperbolas and generates the hyperboloid of one nappe.

The surface whose equation in rectangular coordinates is

$$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$
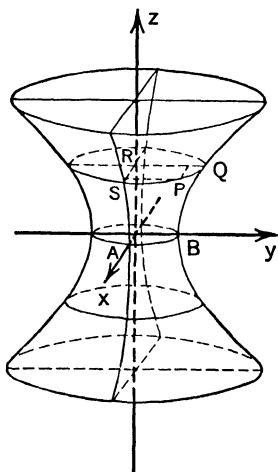
is called the *hyperboloid of two nappes*. Such a quadric surface is represented in Figure 103.

As before, the derivation of this equation may be carried out by allowing a variable ellipse to move with the ends of its axes on the fixed hyperbolas

$$a^2z^2 - c^2x^2 = a^2c^2, \quad y = 0,$$

and $$b^2z^2 - c^2y^2 = b^2c^2, \quad x = 0.$$

At $z = \pm c$, the ellipse has the equation

$$b^2x^2 + a^2y^2 = 0$$

which is the point C or C'. There are no sections of the surface in the interval $-c < z < c$.

If $a = b$, both the hyperboloids of one and two nappes are hyperboloids of revolution.
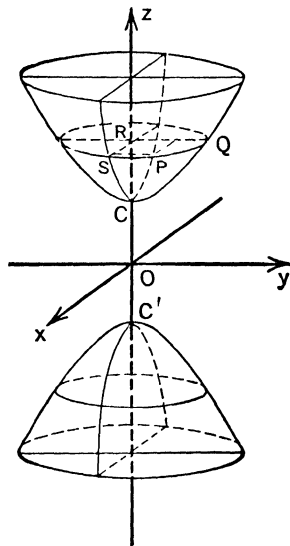
Fig 103

Exercise 87

GROUP A

1. Transform the equations $\rho = 3z$ and $\rho = 4(z - 1)$ to rectangular coordinates and make a drawing of each cone

2. Transform the equations

   $$\phi = \pi/3 \quad \text{and} \quad r\cos\phi - 2 = r\sin\phi$$

   to rectangular coordinates and make a drawing of each cone.

3. Find the traces of the surface $x^2 + y^2 = 4z^2$. Show that sections parallel to the xy-plane are circles and that the sections by planes $x = a$ and $y = b$ are hyperbolas

4. Find the traces and the coordinates of the vertex of the cone $y^2 + z^2 = 4(x - 2)^2$. Show that sections parallel to the yz-plane are circles and that sections by planes $y = b$ and $z = c$ are hyperbolas.

5. Find the traces of the surface $\dfrac{x^2}{4} + \dfrac{y^2}{9} + \dfrac{z^2}{16} = 1$ and find the lengths of the axes and the coordinates of the foci of each trace.

6. Find the equations of the yz-, the xz- and xy-projections of the sections of the ellipsoid given in Problem 5 made by the planes $x = 1$, $y = 2$ and $z = 3$, respectively.

7. Find the traces of the surface $9x^2 - 9y^2 + 4z^2 = 36$ and show that plane sections parallel to the xz-plane are ellipses.

8. Make an analysis of the nature of the sections of the hyperboloid given in Problem 7 by planes $x = a$ for $|a| < 2$, for $|a| = 2$ and for $|a| > 2$.

9. Find the xy- and the xz-traces of the surface $9x^2 - 9y^2 - 4z^2 = 36$ and show that sections by planes $y = b$ and by planes $z = c$ are hyperbolas.

10. Make an analysis of the nature of the sections of the hyperboloid given in Problem 9 by planes $x = a$ for $|a| < 2$ for $|a| = 2$ and for $|a| > 2$.

GROUP B

11. Find the volume enclosed by the conical surface given in Problem 3 from its vertex to the plane $z = 4$

12. Find the volume enclosed by the conical surface given in Problem 4 from the yz-plane to the vertex.

13. Find the volume of the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$, using the fact that the area of an ellipse is π times the product of its semi-axes

14. Find the volume enclosed by the surface given in Problem 7 from the xz-plane to the plane $y = 3$.

15. Find the volume enclosed by the surface given in Problem 9 between the planes $x = 3$ and $x = 4$.

128. Paraboloids.

If a variable ellipse moves with the extremities of its axes on two fixed parabolas which lie in perpendicular planes, which have a common axis and a common vertex and which extend in the same direction, a surface is generated known as an *elliptic paraboloid*. Such a quadric surface is represented in Figure 104.

The equation of the elliptic paraboloid in the figure is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 4cz$$

and may be derived in the usual way, where the equations of the directing parabolas are

$$x^2 = 4a^2cz, \quad y = 0, \quad \text{and} \quad y^2 = 4b^2cz, \quad x = 0.$$

The surface whose equation in rectangular coordinates is

Fig. 104

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 4cz,$$
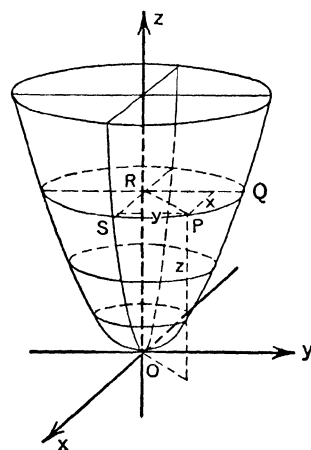
is called the *hyperbolic paraboloid*. Such a quadric surface is represented in Figure 105.

The hyperbolic paraboloid is a *saddle-shaped* surface. In the figure the xz- and the yz-traces are parabolas having a common axis and a common vertex and opening in opposite directions. Also, the plane sections parallel to these coordinate planes are parabolas. The xy-trace of the surface is a pair of lines. The plane sections parallel to the xy-plane are hyperbolas having transverse axes in the yz-plane if above the origin, and having transverse axes in the xz-plane, if below the origin.

129.   The Elliptic Cone.

If a variable ellipse moves with the extremities of its axes on two fixed lines which lie in two perpendicular planes and which intersect on the line
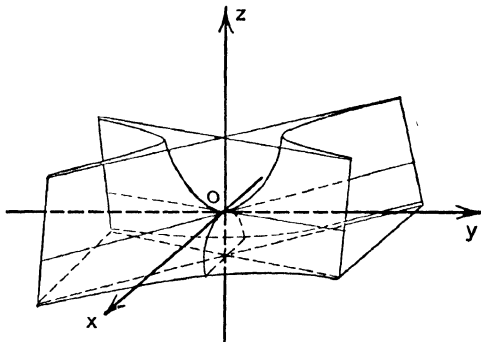
FIG. 105

of intersection of those planes, a surface is generated known as an *elliptic cone*. Such a quadric surface is represented in Figure 106.
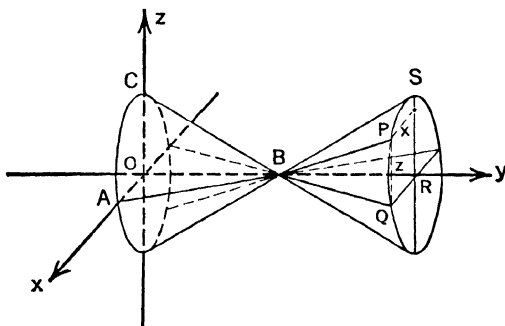
FIG. 106

The equation of the elliptic cone in the figure is

$$\frac{x^2}{a^2} + \frac{z^2}{c^2} = \frac{(b-y)^2}{b^2}$$

and may be derived in the usual way, where the vertex of the cone is the point $B(0,b,0)$ and where the equations of the directing lines are

$$bx + ay = ab, \quad z = 0 \quad \text{and} \quad cy + bz = bc, \quad x = 0.$$

130. Quadric Cylinders.

A variable line parallel to a given line generates a surface known as a *quadric* cylinder if the generatrix is a conic. Such a cylinder is *parabolic, elliptic* or *hyperbolic* according as the conic is a parabola, an ellipse or a hyperbola, respectively. The circular cylinder is a special case of the elliptic cylinder.

The equation of a quadric cylinder can be found if direction numbers of the directrix and the equation of the generatrix are known. The derivation of the equation of the cylinder represented in Figure 107 is as follows:
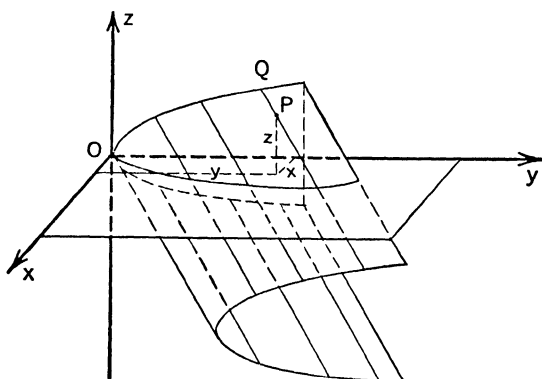
Fig. 107

Let direction numbers of the directrix be a, b and c and let the equation of the generatrix in the yz-plane be the parabola

$$z^2 = 4ky, \quad x = 0.$$

Through any point $P(x,y,z)$ of the surface an element of the surface PQ is drawn parallel to the directrix, where the point $Q(0,q,s)$ is a point of the given parabola. Since this is true, the coordinates of Q satisfy the equation of the parabola,

$$s^2 = 4kq.$$

The equations of the line PQ are

$$\frac{x}{a} = \frac{y - q}{b} = \frac{z - s}{c},$$

from which

$$q = \frac{ay - bx}{a}, \quad s = \frac{az - cx}{a}.$$

Substituting the values of q and s, we have the equation of the parabolic cylinder,

$$(az - cx)^2 = 4ak(ay - bx).$$

131. Transformations of Coordinates.

In some of the more complete treatises on analytic space geometry, a study of the *translation* and the *rotation* of rectangular coordinate axes may be found. Of these, the translation of axes only, is considered here. The purpose of such *transformations* is the simplification of equations of surfaces by referring them to coordinate axes which occupy positions of symmetry. In this way, second degree equations in the variables x, y and z may be reduced to the type forms of the quadrics as given above.
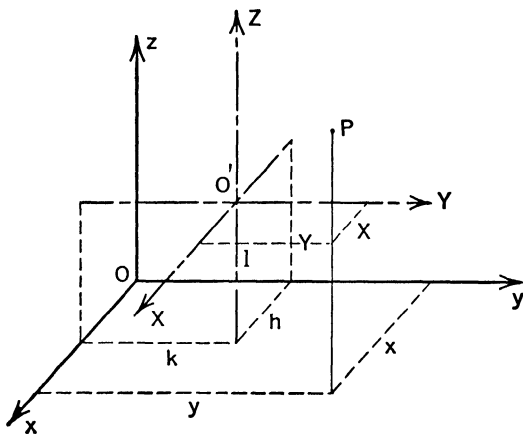
FIG. 108

Let the origin O of the xyz-coordinate system be translated to the point $O'(h,k,l)$ as the origin of an XYZ-coordinate system, in which the pairs of corresponding axes are parallel. From Figure 108, the coordinates of any point P are (x,y,z) or (X,Y,Z), where

$$x = X + h, \quad y = Y + k, \quad z = Z + l.$$

As an example of the simplification of an equation by the translation of axes, consider the equation

$$x^2 - 4y^2 - 6x + 16y - 16z + 9 = 0.$$

Completing the squares in x and y, we have

$$(x^2 - 6x + 9) - 4(y^2 - 4y + 4) = 16z - 9 + 9 - 16,$$

$$(x - 3)^2 - 4(y - 2)^2 = 16(z - 1).$$

Translating axes by letting

$$x = X + 3, \quad y = Y + 2, \quad z = Z + 1,$$

$$\frac{X^2}{4} - \frac{Y^2}{1} = 4Z.$$

Thus the surface is shown to be that of a hyperbolic paraboloid of which the new coordinate axes are line of symmetry. Figure 105 represents such a surface having the same relative position with respect to the coordinate axes.

Exercise 88

GROUP A

Identify and make a sketch showing plane sections of each of the following surfaces.

1. $9x^2 + 36y^2 + 4z^2 = 36.$
2. $9x^2 + 9y^2 - 4z^2 = 36.$
3. $225x^2 - 25y^2 - 9z^2 = 225.$
4. $9x^2 - 4y^2 - 144z = 0.$
5. $4y^2 + z^2 - 32x = 0.$
6. $4z^2 + y^2 - 4 = 0.$
7. $9y^2 - 4z^2 - 36 = 0.$
8. $z^2 - 8x + 8 = 0.$
9. $x^2 + y^2 + z^2 - 2y - 4z = 20.$
10. $4x^2 + 4y^2 + 9z^2 = 36.$
11. $9x^2 + 4y^2 + 9z^2 = 36.$
12. $y^2 + z^2 - 4(x - 1)^2 = 0.$
13. $4x^2 + 9z^2 - 144y^2 = 0.$
14. $x^2 + z^2 + 4y = 0.$
15. $20x^2 - 5y^2 + 4z^2 = 20.$
16. $yz - 4 = 0.$

17. $9x^2 + 4y^2 - 144z = 144.$
18. $\rho = 4z.$
19. $\rho^2 = 4z.$
20. $\rho = 4 - z.$
21. $\rho^2 = z^2 + 4.$
22. $4z^2 - 9\rho^2 = 4.$
23. $4\phi = \pi.$
24. $2\theta = \pi.$
25. $\rho = 10.$
26. $r = 5.$
27. $r \sin \phi = 6.$
28. $\rho \cos \theta = 2.$
29. $\rho(1 + \cos \theta) = 2.$
30. $\rho = 2a \sin \theta.$
31. $r \sin \phi = 2a \cos \theta.$
32. $r \sin \phi = a.$

GROUP B

Transform each of the following equations by translating the coordinate axes to positions of symmetry.

33. $(x - 1)^2 + (y - 2)^2 + (z - 3)^2 = 25.$
34. $\dfrac{(x + 2)^2}{4} + \dfrac{(y + 1)^2}{9} + \dfrac{(z - 1)^2}{16} = 1.$

35. $\dfrac{(y-1)^2}{4} - \dfrac{(x+3)^2}{9} + \dfrac{(z+1)^2}{9} = 1.$

36. $\dfrac{(z-2)^2}{9} - \dfrac{(x-1)^2}{4} - \dfrac{(y+2)^2}{4} = 1.$

37. Find the volume enclosed by the surface given in Problem 33.

38. Find the volume enclosed by the surface given in Problem 34.

39. Find the volume enclosed by the surface $4x^2 + 9y^2 = 36(1 - z)$ above the xy-plane.

40. Find the volume enclosed by the surface $9y^2 + 4z^2 = 36(x - 1)^2$ from the yz-plane to the vertex.

CHAPTER XV

132. Functions of More than One Variable.

This chapter is concerned with a calculus study of functions of more than one variable. While a thorough study of functions of several variables is beyond the scope of a first course in the calculus, a few of the most important definitions and theorems are given, primarily confining attention to functions of two variables.

Functions of more than one variable occur frequently, even in the most elementary mathematics. For example, the area S of a rectangle is a function of its sides,

$$S = xy.$$

Again, the volume V of a rectangular parallelopiped is a function of its three dimensions,

$$V = xyz.$$

In general, if z is an explicit function of x and y, we write

$$z = f(x,y).$$

The symbol for a function of three variables is similarly written,

$$u = f(x,y,z).$$

Continuity. As in the case of functions of a single variable, we shall be concerned chiefly with functions of two variables which are continuous. The definition of the *continuity* of a function of two variables is an extension of the one for a function of one variable given in Section 7.

A function $f(x,y)$ is said to be continuous for $x = a$, $y = b$, if

$$\lim_{\substack{x \to a \\ y \to b}} f(x,y) = f(a,b).$$

In order to be precisely accurate, it is necessary to qualify this definition by saying that the function is defined for the values $x = a$, $x = b$ and for neighboring values, regardless of the manner in which these variables approach their respective limits a and b. Throughout the discussion of

310

functions in this chapter, it is assumed that they are continuous at all points within the range of consideration.

A function of two variables is not always single-valued. For example, if the equation of a sphere,

$$x^2 + y^2 + z^2 = a^2$$

is solved for z, there are two branches of the function.

Thus
$$z = \pm \sqrt{a^2 - x^2 - y^2}.$$

However, as in the case of a function of a single variable, each branch may be treated separately.

If we require a function $f(x,y)$ to be a continuous function, it is the geometric equivalent of saying that we have under consideration a surface

$$z = f(x,y)$$

having no breaks which would make it impossible for a point (x,y,z) moving on the surface to approach the point $[a,b,f(a,b)]$ from any direction or in any manner. If we require, in addition, that the function be single-valued, then for $x = a$, $y = b$, there is one and only one value $f(a,b)$.

133. Partial Derivatives.

A function of two variables can be differentiated with respect to either variable, provided that *the other variable is considered as a constant* during the operation. Such a derivative is the *partial derivative* of the function.

Assume that z is a continuous single-valued function of x and y,

$$z = f(x,y).$$

Then the partial derivatives of the function *with respect to x* and *with respect to y* are defined by

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x,y)}{\Delta x}, \quad \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x,y)}{\Delta y},$$

respectively, where in the first, the variable y is held constant and in the second, the variable x is held constant.

The partial derivatives of z with respect to x and y, may be represented by one of the following symbols:

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x} f(x,y) = f_x(x,y) \quad \text{and} \quad \frac{\partial z}{\partial y} = \frac{\partial}{\partial y} f(x,y) = f_y(x,y),$$

respectively.

Since the partial derivatives of a function of two variables require that one variable be treated as a constant during each differentiation, the formulas for partial differentiation are those for *ordinary* differentiation of a function of a single variable.

Let us find the partial derivatives of the function

$$z = x^2 + xy + y^2.$$

$$\frac{\partial z}{\partial x} = 2x + y, \quad \frac{\partial z}{\partial y} = x + 2y. \quad '$$

The partial derivatives of functions of more than two variables with respect to any one of them may be found by extending the above definition, in which all the other variables, except the one, are considered constant during the process.

134. Geometric Interpretation of Partial Derivatives.

The partial derivatives of a function of two variables have geometric interpretations which are both instructive and useful. Geometrical figures often assist the student to learn that partial differentiation is more than mere technique.

Let $f(x,y)$ be a continuous function and represent the surface $z = f(x,y)$ as in Figure 109. If the variable y is given a constant value, the equation is $y = b$. This is the equation of a plane parallel to the xz-plane.

Hence, the simultaneous equations,

$$z = f(x,y), \, y = b$$

are the equations of the plane curve in which the surface and the vertical plane $PQSB$ intersect. Again, if the variable x is given a constant value, the equation is $x = a$. This is the equation of a plane parallel to the yz-plane. Hence, the simultaneous equations,

$$z = f(x,y), \, x = a,$$

are the equations of the plane curve in which the surface and the vertical plane ATP intersect.

In the figure the surface is cut by the plane $y = b$ in the curve SPQ. At any point $P(x,b,z)$ of the curve, the variable x is given the increment Δx. This locates a point $P'(x + \Delta x, b, z + \Delta z)$ also on the curve. The slope of the secant PP' is

$$\frac{f(x + \Delta x, b) - f(x,b)}{\Delta x} = \tan \alpha.$$

The limit of this difference-quotient as Δx approaches zero is the partial derivative of z with respect to x, for $y = b$. As Δx approaches zero the secant approaches the limiting position, which is the tangent to the curve at the point $P(x,b,z)$. Therefore,

$$\frac{\partial z}{\partial x} = \tan \theta_x,$$

the slope of the curve SPQ at the point P.
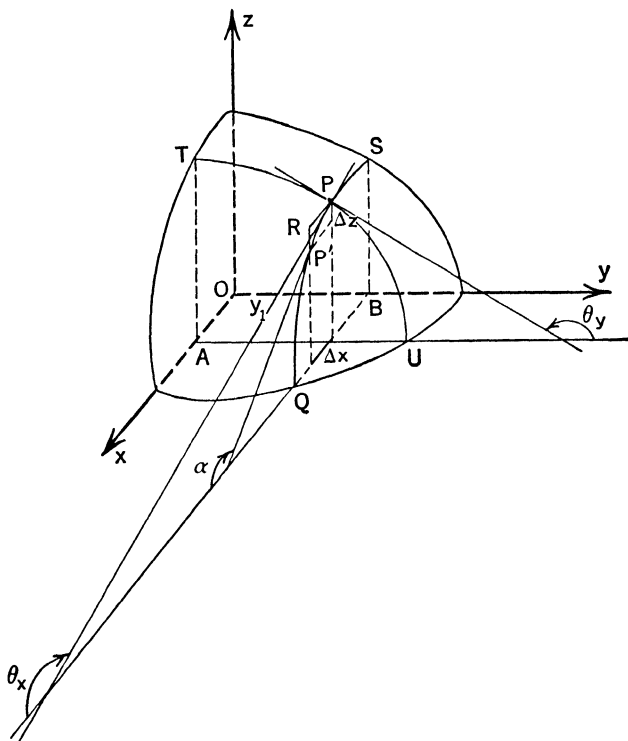
Fig. 109

Similarly, the partial derivative of z with respect to y, for $x = a$, is the slope of the curve TPU at the point $P(a,y,z)$,

$$\frac{\partial z}{\partial y} = \tan \theta_y.$$

Exercise 89

GROUP A

Find the partial derivatives of each of the following functions with respect to x and with respect to y, first solving for z, if necessary.

1. $z = 2x^2 + 3y^2$.

2. $z = 4 - 4x^2 - 9y^2$.

3. $z^2 + y^2 + x^2 = 4$.

4. $z = 3x^2 + 2xy - 2y^2$.

5. $f(x,y) = \dfrac{x}{x - y}$.

6. $z = x^3 + 2x^2y - y^2$.

7. $z = e^{xy}$.

8. $z^2 + x^2 = (y - 1)^2$.

9. Draw a figure representing the first octant portion of the surface $4z = x^2 + 4y^2$ to the plane $z = 4$. Draw the curves of intersection obtained by taking planes parallel to the xz- and the yz-planes through the point $(2,1,2)$. Find the slopes of these curves at the given point.

10. Draw the first octant representations of the curves $x^2 - y^2 + z^2 = 0$, $x = 4$ and $x^2 - y^2 + z^2 = 0$, $y = 5$. Find the slopes of these curves at the point of intersection.

GROUP B

Find the partial derivatives of each of the following functions with respect to each of the variables.

11. $z = xye^{xy}$.

12. $r = \sin \theta \cos \phi$.

13. $r = e^\theta \sin 2\phi$.

14. $z = \ln(x^2 - y^2)$.

15. $z = e^x \sin (x - y)$.

16. $z = \arctan y/x$.

17. $z = c^{x^2 + y^2}$.

18. $z = \arctan \ln (xy)$.

19. Find the coordinates of the point in the first octant on the curve $z^2 = x^2 + y^2$, $x = 3$ at which the slope of its tangent is $\frac{3}{5}$. Also, find the slope of the tangent to the curve of the given surface at this same point, if the curve is cut from the surface by a plane parallel to the xz-plane.

20. Find the slope of each of the curves $x^2 + y^2 + z^2 = 9$, $x = 2$ and $x^2 + y^2 + z^2 = 9$, $y = 1$ at the point $(2,1,2)$. Show that the equations of the tangents to these curves at the given point are $y + 2z = 5$, $x = 2$ and $x + z = 4$, $y = 1$, respectively.

21. Find the equation of the plane tangent to the sphere $x^2 + y^2 + z^2 = 9$ at the point $(2,1,2)$, using the fact that it is perpendicular to the radius at the point of tangency. Show that the two lines whose equations are obtained in Problem 20 lie in this plane.

22. If $u = x^2 + y^2 + xz$, show that $x \dfrac{\partial u}{\partial x} + y \dfrac{\partial u}{\partial y} + z \dfrac{\partial u}{\partial z} = 2u$.

23. If $u = (x - y)(y - z)(z - x)$, show that $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z} = 0$.

24. If $u = x^2 - 4xy + 4y^2$, show that $\dfrac{\partial}{\partial y}\left(\dfrac{\partial u}{\partial x}\right) = \dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial y}\right)$.

135. Tangent Planes.

The equation of the tangent plane at a given point of a surface can be found by means of the partial derivatives of the equation of that surface.

In Figure 110, the tangent plane to the surface $z = f(x,y)$ at the point $P_1(x_1,y_1,z_1)$ on the surface, is determined by the lines P_1Q and P_1R which are tangent to the curves

$$z = f(x,y), \; x = x_1$$

$$z = f(x,y), \; y = y_1,$$

respectively, at their intersection point P_1. The slopes of these lines found from the partial derivatives, evaluated for the coordinates of P_1, are represented by

$$\left(\frac{\partial z}{\partial x}\right)_1, \quad \left(\frac{\partial z}{\partial y}\right)_1.$$
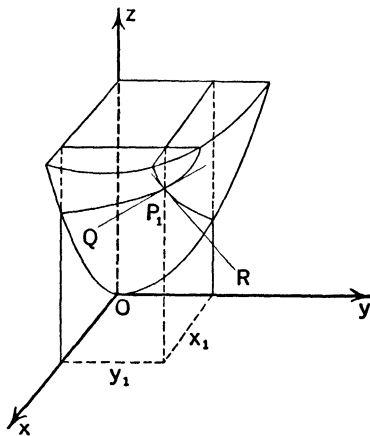
FIG. 110

The equations of the two lines are

$P_1Q:$
$$z - z_1 = \left(\frac{\partial z}{\partial y}\right)_1 (y - y_1), \quad x - x_1 = 0 \quad \text{and}$$

$P_1R:$
$$z - z_1 = \left(\frac{\partial z}{\partial x}\right)_1 (x - x_1), \quad y - y_1 = 0.$$

Hence, their direction numbers are

$$0, \; 1, \; \left(\frac{\partial z}{\partial y}\right)_1 \quad \text{and} \quad 1, \; 0, \; \left(\frac{\partial z}{\partial x}\right)_1,$$

respectively.

The equation of any plane through the point P_1 is

$$A(x - x_1) + B(y - y_1) + C(z - z_1) = 0,$$

where A, B and C are direction numbers of its normal. If the lines P_1Q and P_1R lie in this plane, they are perpendicular to that normal. Hence,

$$A \cdot 0 + B \cdot 1 + C \left(\frac{\partial z}{\partial y}\right)_1 = 0$$

and

$$A \cdot 1 + B \cdot 0 + C \left(\frac{\partial z}{\partial x}\right)_1 = 0,$$

giving

$$\frac{B}{C} = -\left(\frac{\partial z}{\partial y}\right)_1 \quad \text{and} \quad \frac{A}{C} = -\left(\frac{\partial z}{\partial x}\right)_1.$$

Substituting these values in the equation of the plane,

$$z - z_1 = \left(\frac{\partial z}{\partial x}\right)_1 (x - x_1) + \left(\frac{\partial z}{\partial y}\right)_1 (y - y_1),$$

which is the equation of the *tangent plane* to the surface at the given point.

In illustration, we find the equation of the plane tangent to the elliptic paraboloid

$$4z = x^2 + 4y^2$$

at the point $(2,2,5)$. Since

$$\frac{\partial z}{\partial x} = \frac{x}{2}, \quad \left(\frac{\partial z}{\partial x}\right)_1 = 1$$

and since

$$\frac{\partial z}{\partial y} = 2y, \quad \left(\frac{\partial z}{\partial y}\right)_1 = 4.$$

Hence,

$$z - 5 = (x - 2) + 4(y - 2)$$

or

$$x + 4y - z - 5 = 0.$$

136. Higher Partial Derivatives.

The derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ of $z = f(x,y)$, in general, are functions of x and y and are known as the *first partial derivatives*. If a function is differentiated twice with respect to x or y, or once with respect to each x and y, the derivatives are called the *second partial derivatives*. The second partial derivatives of a function are represented by the following symbols:

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} = f_{xx}(x,y), \quad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y\,\partial x} = f_{yx}(x,y),$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x\,\partial y} = f_{xy}(x,y), \quad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2} = f_{yy}(x,y).$$

If $z = f(x,y)$ is a continuous function with continuous derivatives

$$\frac{\partial^2 z}{\partial y\,\partial x} = \frac{\partial^2 z}{\partial x\,\partial y},$$

that is, *the order of differentiation is immaterial.*

Partial derivatives of higher *order* than the second are represented by symbols which are similar to those indicated for the second derivatives. For example, the symbols for the *third partial derivatives* are

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 z}{\partial x^2}\right) = \frac{\partial^3 z}{\partial x^3}, \qquad \frac{\partial}{\partial y}\left(\frac{\partial^2 z}{\partial x^2}\right) = \frac{\partial^3 z}{\partial y\,\partial x^2},$$

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 z}{\partial x\,\partial y}\right) = \frac{\partial^3 z}{\partial x^2\,\partial y}, \qquad \frac{\partial}{\partial y}\left(\frac{\partial^2 z}{\partial y\,\partial x}\right) = \frac{\partial^3 z}{\partial^2 y\,\partial x},$$

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 z}{\partial y^2}\right) = \frac{\partial^3 z}{\partial x\,\partial^2 y}, \qquad \frac{\partial}{\partial y}\left(\frac{\partial^2 z}{\partial y^2}\right) = \frac{\partial^3 z}{\partial y^3},$$

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 z}{\partial y\,\partial x}\right) = \frac{\partial^3 z}{\partial x\,\partial y\,\partial x}, \qquad \frac{\partial}{\partial y}\left(\frac{\partial^2 z}{\partial x\,\partial y}\right) = \frac{\partial^3 z}{\partial y\,\partial x\,\partial y}.$$

As with second derivatives and under the same conditions, the order of differentiation is immaterial. Hence,

$$\frac{\partial^3 z}{\partial x^2\,\partial y} = \frac{\partial^3 z}{\partial y\,\partial x^2} = \frac{\partial^3 z}{\partial x\,\partial y\,\partial x},$$

and

$$\frac{\partial^3 z}{\partial y^2\,\partial x} = \frac{\partial^3 z}{\partial x\,\partial y^2} = \frac{\partial^3 z}{\partial y\,\partial x\,\partial y}.$$

Exercise 90

GROUP A

Find the equation of the plane tangent to each of the following surfaces at the given points.

1. $x^2 + y^2 + z^2 = 17$ at the point $(3, -2, 2)$.

2. $x^2 + y^2 - 4z = 0$ at the point $(2, 2, 2)$.

3. $4x^2 - y^2 + z^2 = 4$ at the point $(-1, 3, 3)$.

4. $9x^2 - 9y^2 - 4z^2 = 36$ at the point $(3, 1, -3)$.

5. $xyz - 8 = 0$ at the point $(1, 2, 4)$.

6. Find the rate of change of the volume of a right circular cone with respect to the radius of its base if the altitude is constant. Find the rate of change of the volume with respect to the altitude if the radius is constant. Find the numerical values of these rates when the radius is 10 ins. and the altitude is 8 ins.

7. Illustrate that $\dfrac{\partial^2 z}{\partial x\,\partial y} = \dfrac{\partial^2 z}{\partial y\,\partial x}$, if $z = \sqrt{x^2 + y^2}$.

8. Show that $b^2\dfrac{\partial^2 z}{\partial x^2} - a^2\dfrac{\partial^2 z}{\partial y^2} = 0$, if $z = \cos ax \cos by$.

9. Show that $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = 0$ if $z = \ln (x^2 + y^2)$.

10. Show that the partial differential equation given in Problem 9 is satisfied if

$$z = \arctan \frac{y}{x}.$$

11. Write the equation of the plane through the point $(2,1,-2)$ parallel to the plane tangent to $x^2 + y^2 + z^2 = 12$ at the point $(2,-2,2)$.

12. At what point on the surface $z = 4x^2 + y^2$ does a tangent line in the plane $y - 1 = 0$ have a slope of 2? Write the equations of the tangent plane at this point and the plane through the origin parallel to it.

GROUP B

Write the equation of the plane tangent to each of the following surfaces at the point (x_1, y_1, z_1) on each surface.

13. $x^2 + y^2 + z^2 = a^2$.
14. $Ax^2 + By^2 + Cz^2 = F$.
15. $b^2c^2x^2 - a^2c^2y^2 \pm a^2b^2z^2 = a^2b^2c^2$.
16. $b^2x^2 \pm a^2c^2y^2 = 4pa^2b^2z$.

17. Find the four second partial derivatives of z with respect to x and y, if

$$z = (x^2 + y^2) \arctan \frac{y}{x}$$

18. Show that the second partial derivative of the function $\sin (x - y) + \ln (x + y)$ with respect to x is equal to the second partial derivative with respect to y.

19. Find the equation of the line tangent to the curve $z = x^2 + 2y^2$, $x = 2$ at the point $(2,1,6)$.

20. Find the three first partial derivatives of the function $z = xy \cos \theta$.

21. The variation of the volume v of a certain gas, the pressure p and the temperature t is given by the equation $pv = kt$, where k is a constant. Find the rates of change of each variable with respect to each variable when the third is held constant.

22. The sides of a triangle are x, y and z and the angle opposite the side z is θ. Find the rates of change of z with respect to x, if y and θ are held constant, with respect to y, if x and θ are held constant and with respect to θ, if x and y are held constant.

23. Show that $x^2 \dfrac{\partial^2 u}{\partial x^2} + y^2 \dfrac{\partial^2 u}{\partial y^2} + z^2 \dfrac{\partial^2 u}{\partial z^2} - 4u = 0$, if $u = \dfrac{x + y + z}{xyz}$.

24. If $f(x,y,z) = x^2y^2 + x^2z^2 + y^2z^2$, show each of the following:

$$f_{xy} = f_{yx}, \quad f_{xz} = f_{zx}, \quad f_{yz} = f_{zy}, \quad f_{xyz} = f_{yzx} = f_{zyx} = 0.$$

25. If $f(x,y) = x \sin y + y \cos x$, show that $f_{xxy} = f_{xyx} = f_{yxx}$.

137. Total Differentials.

For the function, $z = f(x,y)$, if x and y are given the *independent* increments Δx and Δy, respectively, the increment of the function is

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x,y).$$

In general, such an increment is made up of infinitesimals of the first order and higher. For example, in the function

$$z = 2x^2 - xy + y^2,$$
$$\Delta z = 2(x + \Delta x)^2 - (x + \Delta x)(y + \Delta y) + (y + \Delta y)^2 - z$$
$$= (4x - y)\,\Delta x + (2y - x)\,\Delta y + (2\overline{\Delta x}^2 - \Delta x\,\Delta y + \overline{\Delta y}^2).$$

The *principal part* of Δz is called the *total differential of z* and is represented by dz. In the example,

$$dz = (4x - y)\,\Delta x + (2y - x)\,\Delta y.$$

But since

$$\frac{\partial z}{\partial x} = 4x - y, \qquad \frac{\partial z}{\partial y} = 2y - x,$$

the differential of z may be written in terms of these partial derivatives,

$$dz = \frac{\partial z}{\partial x}\,\Delta x + \frac{\partial z}{\partial y}\,\Delta y.$$

In Chapter III the differential of a function of one variable was expressed in terms of dx instead of Δx. In the same way here, in the interest of symmetry, we shall represent Δx by dx and Δy by dy when expressing the differential dz of a function of the independent variables x and y. In general, for any continuous function $z = f(x,y)$ having continuous derivatives, it can be shown that

$$dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy.$$

If x and y are functions of a third variable t, then z is a function of the single variable t and the latter expression reduces to

$$dz = \frac{dz}{dt}\,dt.$$

This is in accord with the definition given for a differential of a function of one variable in Section 24. This is as far as the justification of the form given for the differential of a function of two variables is carried in this book.

In the expression for the total differential of a function,

$$dz = d_x z, \quad \text{if} \quad dy = 0$$

and

$$dz = d_y z, \quad \text{if} \quad dx = 0,$$

where $d_x z$ and $d_y z$ are called the *partial differentials* of z. Since dx and dy are independent differentials, the total differential of a function is the sum of the partial differentials of z with respect to x and y,

$$dz = d_x z + d_y z.$$

If

$$z = \frac{x}{y}, \quad d_x z = \frac{1}{y} dx \quad \text{and} \quad d_y z = -\frac{x}{y^2} dy.$$

Hence,

$$dz = \frac{y\, dx - x\, dy}{y^2}.$$

The expressions for the total differentials of functions of more than two variables are defined by extensions of the case of a function of two variables. For a function

$$u = f(x,y,z),$$

the total differential is *defined* by the expression

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz,$$

and similarly, for a function of a larger finite number of independent variables. Such formulas may be used as the principal part of the increment of the given function for small independent increments of the several variables.

The above derivation assumed that $f(x,y)$ was a function of two independent variables and was extended to a function of three independent variables x, y and z. In more advanced treatises it is shown that these results hold when the variables are dependent.

138. Approximations.

The total differential and the increment of a function of two variables differ by a small amount for *small* values of the independent increments of those two variables. Consequently, the differential of a function may be used often to calculate approximations of changes of values of functions which are produced by small changes in the independent variables.

If two sides and the included angle of a triangle are measured as 30 ins., 40 ins. and 30° and if the first side has a possible error of 0.005 in. and the second, 0.01 in., let us find an approximation of the largest possible error in the area of the triangle due to the errors of measurement.

Let x and y represent the first and second sides, respectively. Then the area is

$$S = \tfrac{1}{4}xy.$$

The approximate error in S is

$$dS = \tfrac{1}{4}(y\,dx + x\,dy)$$

$$= \frac{40(0.005) + 30(0.01)}{4} = 0.125 \text{ sq. ins.,}$$

in which dx and dy are taken with the same sign, since the largest possible error is required. The approximate relative error is

$$\frac{dS}{S} = \frac{0.125}{300} = 0.000417.$$

Exercise 91

GROUP A

Given the function $z = x^2 + xy + y^2$.

1. Find Δz corresponding to the increments Δx and Δy.

2. Find dz corresponding to the differentials dx and dy.

3. Write the expression for $\Delta z - dz$ as a function of the increments of x and y.

4. Find Δz and dz as x changes from 2 to 2.02 and y changes from 1 to 1.01.

5. Using the increment of z, compute the exact value of z for $x = 2.02$ and $y = 1.01$.

6. Using dz, compute the approximate value of z for $x = 2.02$ and $y = 1.01$.

7. Approximate the value of the function for $x = 3.001$ and $y = 2.0006$.

Find the total differentials of each of the following functions.

8. $z = x^3 - 4xy + 3y^2$.　　　　　　**10.** $z = \arcsin(y/x)$.

9. $z = \dfrac{x + y}{x - y}$.　　　　　　**11.** $z = e^{xy}$.

12. A rectangular metal box 8 ins. high has a base 10 ins. square. If the metal is 0.05 in. thick, compute the approximate volume of metal.

Compute the approximate values of each of the following.

13. $(4.002)^2(2\,001)^3$, using $z = x^2y^3$.

14. $\sqrt{(8.002)(1.99)}$, using $z = \sqrt{xy}$.

15. Find the approximate largest error in computing the value of y/x^2 if $y = 100$ with a possible error of ±0.02 and if $x = 2$ with a possible error of ±0.1. To find the largest error dx and dy should be taken of opposite signs.

16. Find the approximate relative error and the approximate relative percentage error in the computation in Problem 15.

GROUP B

17. Find the increment and the differential of the function $\ln \dfrac{x^2 - y^2}{xy}$ for $x = 2$, $\Delta x = 0.02$, $y = 1$ and $\Delta y = 0.01$.

Find the approximate value of each of the following.

18. $\sqrt{(7 - 0.02)^2 + (24.04)^2}$, using $\sqrt{x^2 + y^2}$.

19. $(1.01)(1.02) \ln (1.01)(1.02)$, using $xy \ln xy$.

20. $\dfrac{1}{4.04} \sin \dfrac{\pi}{(3.02 - 1.02)}$, using $\dfrac{1}{x + y} \sin \dfrac{\pi}{x - y}$.

21. The equal sides of an isosceles triangle are 10 ins. long and the base is 8 ins. Find the approximate greatest error committed in computing the area of the triangle caused by errors of 0.02 in. in the length of each of the equal sides and 0.03 in. in the length of the base.

22. By measurement two sides of a triangle are found to be 20 ins. and 15 ins. and the included angle 60°. If there is a possible error of 0.02 in. in measuring the length of each of the sides and of 1° in measuring the angle, find the greatest possible error committed in computing the area of the triangle.

139. Implicit Differentiation.

If z is a variable which is an implicit function of the two independent variables x and y, the equation may be written $F(x,y,z) = 0$. The partial derivatives of z with respect to x and with respect to y have been found thus far by first expressing z as an explicit function of x and y. This process is called *explicit partial differentiation*. The process whereby it is possible to find these derivatives from the implicit form of an equation is called *implicit partial differentiation*.

Let y be an implicit function of x which is defined by the equation

$$f(x,y) = 0,$$

and let

$$z = f(x,y)$$

for the moment. Then from the previous section,

$$dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy.$$

But since $z = 0$ and $dz = 0$, we have

$$\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = 0$$

and

$$\frac{dy}{dx} = - \frac{\dfrac{\partial f}{\partial x}}{\dfrac{\partial f}{\partial y}}, \quad \text{provided that } \frac{\partial f}{\partial y} \neq 0.$$

This value of the derivative of an implicit function of two variables conforms with that which was found in Section 37.

Analogously, we shall consider an implicit function of three variables. Let z be an implicit function of the two independent variables x and y defined by the equation

$$F(x,y,z) = 0,$$

and let

$$u = F(x,y,z)$$

for the moment. Then from the previous section,

$$du = \frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial y}\,dy + \frac{\partial F}{\partial z}\,dz.$$

But since $u = 0$ and $du = 0$, we have

$$\frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial y}\,dy + \frac{\partial F}{\partial z}\,dz = 0$$

First, if we hold y fixed, $dy = 0$ and we have the first equation below. Second, if we hold x fixed, $dx = 0$ and we have the second equation below.

$$\frac{\partial z}{\partial x} = -\frac{\dfrac{\partial F}{\partial x}}{\dfrac{\partial F}{\partial z}}, \quad \frac{\partial z}{\partial y} = -\frac{\dfrac{\partial F}{\partial y}}{\dfrac{\partial F}{\partial z}}, \quad \text{provided that } \frac{\partial F}{\partial z} \neq 0.$$

Suppose that we wish to write the equation of the plane tangent to the surface $x^2 + xy + y^2 + z^2 + xz - 19 = 0$ at the point $(1,2,3)$. We proceed as follows:

$$(2x + y + z) + (2z + x)\frac{\partial z}{\partial x} = 0, \quad \frac{\partial z}{\partial x} = -\left.\frac{2x + y + z}{2z + x}\right]^{(1,2,3)} = -1.$$

$$(x + 2y) + (2z + x)\frac{\partial z}{\partial y} = 0, \quad \frac{\partial z}{\partial y} = -\left.\frac{x + 2y}{2z + x}\right]^{(1,2,3)} = -\frac{5}{7}.$$

$$z - 3 = -(x - 1) - \tfrac{5}{7}(y - 2).$$

$$7x + 5y + 7z - 38 = 0.$$

Exercise 92

1. If $z = x^2 + y^2 - a^2 = 0$, show that $\dfrac{dy}{dx} = -\dfrac{\dfrac{\partial z}{\partial x}}{\dfrac{\partial z}{\partial y}}$.

2. If $f(x,y) = \ln(y/x) + ye^x = 0$, find $\dfrac{dy}{dx}$, using the partial derivatives of the given function.

Find the equation of the tangent to each of the following surfaces at the given points.

    3. $x^2 + y^2 + z^2 = 49$ at the point $(3,2,-6)$.
    4. $3x^2 + 2y^2 + z^2 = 30$ at the point $(2,1,4)$.
    5. $2x^2 + y^2 - z^2 = 0$ at the point $(2,1,3)$.
    6. $x^2 + y^2 + z^2 - 4x - 2y - 6z = 11$ at the point $(2,1,-2)$.

Find the partial derivatives of z with respect to x and y for each of the following functions.

    7. $xz - \cos yz = a$.
    8. $(x - y)(x - z)(y - z) = c$.
    9. $e^{xy} + e^{xz} + e^{yz} = A$.
    10. $e^{xy} \sin z - e^z \sin (xy) = 0$.

11. Show that the surfaces $x^2 + 3y^2 + 2z^2 = 9$ and $x^2 + y^2 + z^2 - 8x - 8y - 6z + 24 = 0$ are tangent to each other at the point $(2,1,1)$.

12. Find the equation of the plane through the point $(-2,1,-2)$ which is parallel to the plane tangent to the surface $x^2y^2 + 2x + z^3 - 16 = 0$ at the point $(2,1,2)$.

CHAPTER XVI

140. Multiple Integration.

In Chapter X a definite double integral was defined and used in finding plane areas and volumes of revolution. In this section it is desired to discuss both definite and indefinite multiple integrals in order to relate the processes of partial differentiation and integration.

The symbol for an indefinite single integral of a function of one variable with respect to that variable has been used frequently in this book. It is

$$\int f(x)\ dx,$$

which represents any function of x whose first derivative is $f(x)$. In the same manner, the symbol

$$\int \int f(x)\ dx^2,$$

represents any function of x whose second derivative is $f(x)$. It is called an indefinite iterated integral of $f(x)$ with respect to x.

The symbol

$$\int \int f(x,y)\ dx\ dy$$

is an indefinite iterated integral of $f(x,y)$ with respect to x and y. It indicates that the given function is to be integrated with respect to x while y is held constant and that the result is then to be integrated with respect to y. The first integration is a *partial integration* and is the reverse process to that of partial differentiation.

Continued partial integration n times of a function of n variables with respect to each of them is called indefinite *multiple integration*. The symbols representing such integrals are called *multiple integrals*.

Irrespective of the order of integration, a multiple integral of a function of $x,y,z,\cdots$, with respect to each variable, is another function whose mixed partial derivatives of the same order of multiplicity, is the given function.

Analogously to the evaluation of an indefinite integral of a function of one variable between two limits giving rise to a single definite integral, the evaluation of a multiple integral of a function of several variables gives rise to a *definite multiple integral*. The symbol

$$\int_a^b \int_{y_1}^{y_2} \int_{z_1}^{z_2} f(x,y,z) \; dz \; dy \; dx,$$

is called, for brevity, a definite integral. It indicates that three successive integrations are to be carried out and *in a definite order*. The limits z_1 and z_2 are constants or functions of x and y, the limits y_1 and y_2 are constants or functions of x and the limits a and b are constants. During the integration with respect to z, the variables x and y are regarded as constants and during the integration with respect to y, the variable x is regarded as a constant.

141. Volume under a Surface.

The methods which have been used in previous chapters in finding the volumes of solids were applicable to limited classes of solids, such as those obtained by the revolution of a plane area about an axis. In addition, the volumes of a few other solids were obtained in Section 53, where the area of a cross section was known. In this chapter these methods are extended and are applied to the problem of finding the volumes of solids of a more general nature. These methods employ multiple integrals which are applied, not only to problems of finding volumes, but also to other problems: moments of mass, moments of inertia, centroids and attraction.

In this section double integration is applied to the problem of finding volumes of right cylindrical-shaped solids. Such a solid is bounded by an enclosed area of a given surface, the projection of that area on one of the coordinate planes and the cylindrical surfaces of projection. This problem is often spoken of as that of finding the *volume under a surface*. In addition, the application of the method permits the solution of the problem of finding a cylindrical-shaped volume between two surfaces.

In Figure 111 the area $MNQR$ is assumed to lie on the surface

$$z = f(x,y),$$

where $f(x,y)$ is a continuous single-valued function over the area $EFTS$ on the xy-plane. It is also assumed that MN and QR are segments of the curves of intersection of the given surface with the cylindrical surfaces

$$y = g_1(x) \quad \text{and} \quad y = g_2(x),$$

respectively, and that NQ and MR are segments of the curves of intersection of the given surface with the planes

$$x = a \quad \text{and} \quad x = b,$$

respectively. Consequently, the plane area $EFTS$ is the projection of the area $MNQR$ on the xy-plane.

The line segment AB is divided into m equal parts, each equal to Δx, and at each point of division, lines are drawn parallel to the y-axis. Then, a set of lines is drawn in the xy-plane parallel to the x-axis at equal distances Δy between adjacent lines. In this manner, a set of elementary rectangles of dimensions Δx by Δy, is formed on the xy-plane. One such rectangle is drawn in the figure at the point $P'(x,y,0)$.
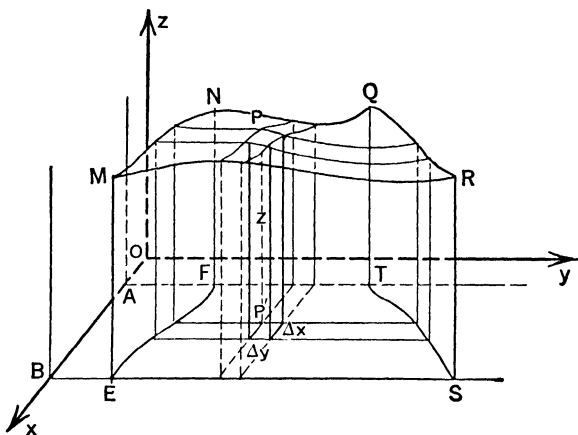
FIG. 111

At each vertex of the elementary rectangles within the area $EFTS$, a line is drawn parallel to the z-axis and terminated by the given surface. Since the point $P(x,y,z)$ lies on the surface, the line segment $P'P = z$. In this manner, there is formed a set of *right rectangular prisms* which are the elements of volume. If $P'(x_i,y_j,0)$ represents a point within the area $EFTS$ on the jth parallel to the x-axis and on the ith parallel to the y-axis, the volume of the corresponding elemental prism is

$$dV = z \, \Delta x \, \Delta y = f(x_i,y_j) \, \Delta x \, \Delta y.$$

For sufficiently small values of Δx and Δy, this element is an approximation to the increment of volume, ΔV, having the rectangle Δx by Δy as base,

having vertical edges and having as its top the intercepted portion of the given surface.

Holding x_i and Δx fixed, the sum

$$\sum_{j=1}^{j=n} \left[f(x_i,y_j) \, \Delta y \right] \Delta x,$$

is approximately the volume of the solid included between the ith and the $(i + 1)$st planes parallel to the yz-plane. The sum of the volumes of all such *slices* forms an approximation to the volume between the planes $x = a$ and $x = b$. The limit of this sum, as the numbers of planes parallel to the the xz- and the yz-planes becomes infinite, is the required volume. Thus,

$$V = \lim_{m,n \to \infty} \sum_{j=1}^{j=n} \sum_{i=1}^{i=m} f(x_i,y_j) \, \Delta y \, \Delta x.$$

By the fundamental theorem for functions of two variables, mentioned in Section 93,

$$V = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y) \, dy \, dx.$$
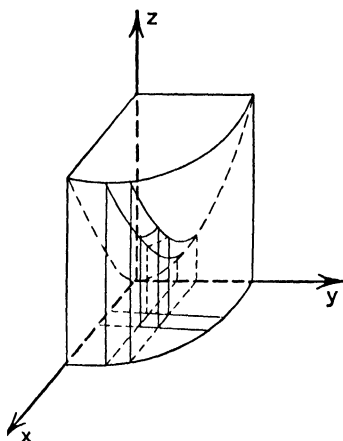
In the foregoing discussion, the cylindrical-shaped solid under the plane was divided into elements of volume which are right prisms with edges parallel to the z-axis. Frequently, it is more convenient and sometimes necessary, to choose elements of volume whose edges are parallel to one of the other coordinate axes. In each application, a sketch of the required volume should be made and by means of it, the order of integration determined and the element of volume expressed.

Let it be required to find the volume below the surface $z = x^2 + y^2$, above the xy-plane and within the cylinder $x^2 + y^2 = a^2$.

FIG. 112

The paraboloid and the cylinder are represented in Figure 112, where a vertical element of volume is constructed. The volume of this element is

$$dV = z \, \Delta x \, \Delta y = (x^2 + y^2) \, \Delta x \, \Delta y.$$

Hence,

$$V = 4 \int_0^a \int_0^{\sqrt{a^2-y^2}} (x^2 + y^2)\, dx\, dy$$

$$= 4 \int_0^a \left[xy^2 + \frac{x^3}{3} \right]_0^{\sqrt{a^2-y^2}} dy$$

$$= \frac{4}{3} \int_0^a \left[3y^2 \sqrt{a^2 - y^2} + (a^2 - y^2)\sqrt{a^2 - y^2} \right] dy$$

$$= \frac{4}{3} \int_0^a (2y^2 \sqrt{a^2 - y^2} + a^2 \sqrt{a^2 - y^2})\, dy$$

$$= \frac{1}{3} \left[-2y(a^2 - y^2)^{3/2} + 3a^2 y \sqrt{a^2 - y^2} + 3a^4 \arcsin \frac{y}{a} \right]_0^a$$

$$= \frac{\pi}{2}\, a^4 \text{ cubic units.}$$

If a volume between two surfaces is to be found, we shall consider only those cases in which the element of volume may be expressed as the difference between two right prisms having the same base but different altitudes. Such a case is illustrated by the following solution in which it is to be noted that the equation of the projection of the curve of intersection of the two surfaces must be known.

To find the volume enclosed by the surfaces

$$z = x^2 + y^2 \quad \text{and} \quad z = 8 - x^2 - y^2,$$

use is made of Figure 113. The surfaces intersect in a circle one-fourth of which is the arc $A(2,0,4)\, B(0,2,4)$. The equation of the projection of this circle on the xy-plane is $x^2 + y^2 = 4, z = 0$.

We let z' and z'' represent the z-coordinates of the points on the first and the second surface, respectively, which are vertically above any point $P(x,y,0)$ within the circular arc $A'B'$. Then the volume of an element between the two surfaces is

Fig. 113

$$dV = (z'' - z')\, \Delta x\, \Delta y = 2(4 - x^2 - y^2)\, \Delta x\, \Delta y.$$

Taking the limit of the sum of all such elements and applying the funda-

mental theorem, we have
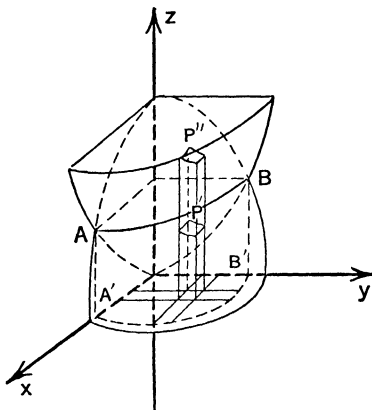
$$V = 8 \int_0^2 \int_0^{\sqrt{4-y^2}} (4 - x^2 - y^2) \, dx \, dy,$$

in which four times the volume in the first octant has been taken. From this

$$V = \frac{8}{3} \int_0^2 \left[12x - x^3 - 3xy^2 \right]_0^{\sqrt{4-y^2}} \, dy = 16\pi.$$

Exercise 93

GROUP A

Evaluate each of the following integrals.

1. $\int_0^2 \int_0^3 xy(x - y) \, dy \, dx.$

2. $\int_1^2 \int_0^y x \, dx \, dy.$

3. $\int_0^1 \int_{\sqrt{y}}^{2-y} y^2 \, dx \, dy.$

4. $\int_0^1 \int_{x^2}^{2-x} y^2 \, dy \, dx.$

Find the volume of each of the following solids bounded as specified.

5. Bounded by $3x + 2y + 4z = 12$ and the coordinate planes.

6. Bounded by $2x - 3y + 6z = 12$ and the coordinate planes, by taking elemental prisms with edges parallel to the y-axis.

7. Bounded by $4z = 16 - 4x^2 - y^2$ and $z = 0$.

8. Bounded by $z = 4 - x^2 - y^2$ and the xy-plane.

9. Bounded by $x^2 + y^2 - 4x = 0$, $z - x = 0$ and the xy-plane.

10. Under the surface $z = 16 - x^2 - y^2$ above the xy-plane and between the cylinders $x^2 = 4y$ and $y^2 = 4x$.

GROUP B

Find the volume of each of the following solids bounded as specified.

11. Bounded by $x = 9 - y^2 - z^2$ and $x = 0$.

12. Bounded by $z = x^2 + y^2$, $x + y = 2$ and the coordinate planes.

13. Bounded by $2z = x^2 + y^2$ and $x^2 + y^2 = 4$.

14. Bounded by $y^2 = 4 - 2z$, $x^2 + y^2 = 4$ and $z = 0$.

15. Bounded by $y^2 = 4 - x$, $z = x^2 + y^2$, $z = 0$ and $x = 0$.

16. The first octant volume bounded by $z = x + y^2$ and $x = 4 - y^2$.

17. Common to the cylinders $y^2 + z^2 = a^2$ and $x^2 + z^2 = a^2$.

18. The first octant volume between the xy-plane and the surface $xz = 4$, bounded by the surface $xy = 4$, $x = 4$ and $y = 4$.

19. Under the surface $z = 12 + y - x^2$ above the xy-plane and between the cylinders $y^2 = x$ and $x^2 = y$.

20. Bounded by the surface $x^{2/3} + y^{2/3} + z^{2/3} = a^{2/3}$.

142. Double Integration in Cylindrical Coordinates.

In many problems in which a cylindrical-shaped volume is required, the task of evaluating the integrals is simplified by the use of cylindrical co-

ordinates. This is particularly true if the bounding surfaces are coaxial with one of the coordinate axes.

Consider the volume of a solid bounded by the surface

$$z = f(\rho,\theta),$$

the $\rho\theta$-plane, the cylindrical surfaces

$$\rho = g_1(\theta) \quad \text{and} \quad \rho = g_2(\theta)$$

and the planes

$$\theta = \alpha \quad \text{and} \quad \theta = \beta.$$

The volume is divided into elements by means of n vertical planes through the z-axis and m vertical coaxial cylinders about the z-axis. The area of a cross section of such an element is equal to

$$dS = \rho \, \Delta\rho \, \Delta\theta,$$

as shown in Section 113. Hence, the element of volume is $z \, dS$,

$$dV = z\rho \, \Delta\rho \, \Delta\theta = \rho f(\rho,\theta) \, \Delta\rho \, \Delta\theta.$$

Taking the limit of the sum of all such elements as m and n become infinite, and applying the fundamental theorem, the volume is

$$V = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} \rho f(\rho,\theta) \, d\rho \, d\theta.$$

To illustrate the comparative simplicity of the use of cylindrical coordinates in certain instances, the volume of the solid described in the last section is found. It is required to find the volume below the parabolic surface $z = \rho^2$, above the $\rho\theta$-plane and within the cylinder $\rho = a$.
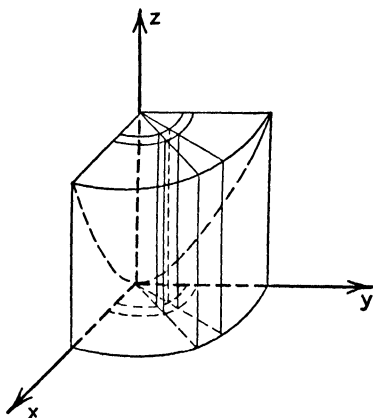
FIG. 114

An element of volume is represented in Figure 114. The volume of this element is

$$dV = z\rho \, \Delta\rho \, \Delta\theta = \rho^3 \, \Delta\rho \, \Delta\theta.$$

Hence, the volume is

$$V = 4 \int_0^{\pi/2} \int_0^a \rho^3 \, d\rho \, d\theta = \int_0^{\pi/2} \rho^4 \Big]_0^a \, d\theta$$

$$= \int_0^{\pi/2} a^4 \, d\theta = a^4 \theta \Big]_0^{\pi/2} = \frac{\pi}{2} a^4 \text{ cu. units.}$$

As a second illustration, let us find the volume of the cylinder $\rho = 2a \sin \theta$ included between the planes $z = a$ and $z = 2a - \rho \cos \theta$.
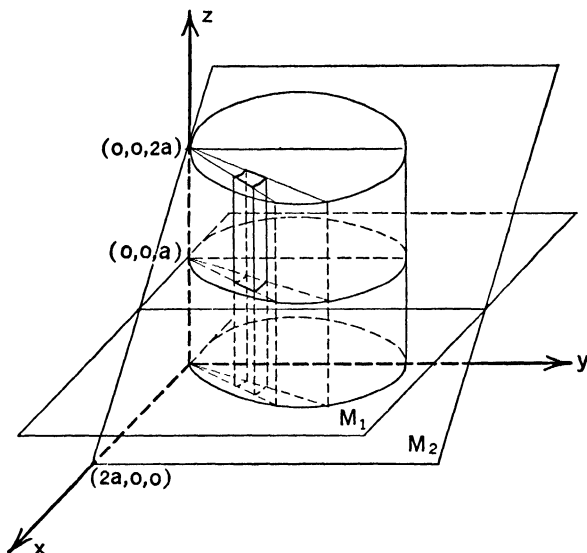
Fɪɢ. 115

The length of the element between the planes M_1 and M_2, as represented in Figure 115 is $z'' - z'$. Hence, the volume of the element is

$$dV = \rho(z'' - z') \, \Delta\rho \, \Delta\theta = \rho[(2a - \rho \cos \theta) - a] \, \Delta\rho \, \Delta\theta.$$

The required volume of the solid·is

$$V = \int_0^\pi \int_0^{a \sin \theta} (a\rho - \rho^2 \cos \theta) \, d\rho \, d\theta$$

$$= \int_0^\pi \left[\frac{a}{2} \rho^2 - \frac{1}{3} \rho^3 \cos \theta \right]_0^{a \sin \theta} d\theta$$

$$= a^3 \int_0^\pi (\tfrac{1}{2} \sin^2 \theta - \tfrac{1}{3} \sin^3 \theta \cos \theta) \, d\theta$$

$$= \frac{a^3}{24} \left[6\theta - 3 \sin 2\theta - 2 \sin^4 \theta \right]_0^\pi = \frac{\pi}{4} a^3 \text{ cu. units.}$$

Exercise 94

GROUP A

1. Find the volume of a right circular cylinder of radius a and height h, using cylindrical coordinates.

Find the volume of each of the following solids bounded as specified.

2. Bounded by $\rho = 2a \cos \theta$, $z = \rho \cos \theta$ and $z = 0$.
3. Cut from the sphere $\rho^2 + z^2 = 4a^2$ by $\rho = 2a \cos \theta$.
4. Above the surface $\rho^2 = az$ and below $\rho^2 + z^2 = 2az$.
5. Above the plane $z = 0$, inside the cylinder $\rho = a$ and under the cone $\rho = z$.
6. Enclosed by the surface $\rho^2 + z^2 = a^2$.
7. In the first octant outside $\rho = z$ and inside $\rho = a \sin 2\theta$.
8. Cut from the sphere $\rho^2 + z^2 = a^2$ by $\rho = a \sin \theta$.

GROUP B

9. Find the volume common to the sphere $x^2 + y^2 + z^2 = 4a^2$ and the cylinder $z^2 + y^2 = a^2$ by cylindrical coordinates.
10. Find the volume between the planes $z = 0$ and $z = \rho \cos \theta$ and within the cylinder $\rho = a \cos 2\theta$.
11. If the curve $az^2 = y^3$ is rotated about the z-axis, find the volume between this surface and the surface $\rho = a$
12. Find the volume between $z = 0$ and the cylinders $y^2 = a^2 - az$ and $\rho^2 = a^2 \cos 2\theta$.

143. Volumes by Triple Integration.

The symbol for a *triple definite integral* of a function of three variables may be written

$$\int_a^b \int_{y_1}^{y_2} \int_{z_1}^{z_2} F(x,y,z) \; dz \; dy \; dx.$$

First, holding x and y constant, the inner integral is evaluated. Since, in general, the inner limits are functions of x and y, the integral is a function of x and y. Thus, we have

$$\int_a^b \int_{y_1}^{y_2} G(x,y) \; dy \; dx.$$

Next, holding x constant, the second integral is evaluated. Since the limits of this integral, in general, are functions of x, the integral is a function of x. Thus, we have

$$\int_a^b Q(x) \; dx.$$

Finally, the evaluation of this integral yields a numerical result.

In this section the volumes of solids are computed by means of triple integration.

If the equations of the surfaces which bound the solid whose volume is sought are given in rectangular coordinates, we may formulate the volume by means of a triple integral as follows:

Three sets of parallel planes are constructed. One set of planes is taken parallel to the yz-plane so that any two adjacent ones are separated by the distance Δx, one set is taken parallel to the xz-plane so·that any two adjacent ones are separated by the distance Δy and one set is taken parallel to the xy-plane so that any two adjacent ones are separated by the distance Δz. These planes divide the solid into small rectangular *blocks*, each having the volume

$$dV = \Delta x \, \Delta y \, \Delta z,$$

together with the irregular portions around the boundaries. The sum of the volumes of all such elemental blocks which lie wholly within the solid is an approximation to the required volume. The limit of this sum is the volume of the solid, if the dimensions of the blocks approach zero as limits. Thus,

$$V = \lim_{l,m,n \to \infty} \sum_{i=1}^{i=l} \sum_{j=1}^{j=m} \sum_{k=1}^{k=n} \Delta z_k \, \Delta y_j \, \Delta x_i.$$

In this notation, $\Delta z_k \, \Delta y_j \, \Delta x_i$ denotes the volume of an element which has a vertex at a point $P(x_i, y_j, z_k)$ and which is included between the kth and the $(k+1)$st planes parallel to the xy-plane, between the jth and the $(j+1)$-st planes parallel to the xz-plane and between the ith and the $(i+1)$st planes parallel to the yz-plane. The limit of this triple sum is expressed as a triple integral whose limits are obtained from the given boundaries of the solid. More advanced treatises show that the limit of this sum can be evaluated by successive integration as has been indicated.

To find the volume of the solid bounded by the paraboloid and the cylinder,

$$z = x^2 + y^2, \quad z = 4 - 8y^2,$$

by triple integration, we represent the first octant portion of the solid in Figure 116. The volume of an element of the solid is

$$dV = \Delta x \, \Delta y \, \Delta z.$$

It is convenient, in this case, to integrate with respect to z first, since the upper and lower limits for z are readily expressed as functions of x and

y. In fact, they are the functions given above. If the second integration is taken with respect to y, the upper limit is found by the elimination of z from the two given equations. Thus,

$$y = \frac{\sqrt{4 - x^2}}{3}.$$

Since the xy-plane forms the lower boundary of the solid, the lower limit of y is zero. The final integration must be performed with respect to x with constant limits. The upper limit of x is found from the latter equation for $y = 0$, since it is the xy-projection of the curve of intersection of the two given surfaces. The lower limit of x is zero, since we wish to express one fourth of the total volume. Thus,

$$x = 2 \quad \text{and} \quad x = 0.$$
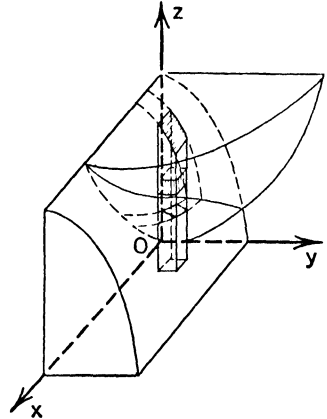
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The solution for the required total volume above the xy-plane is carried out as follows:

$$V = 4 \int_0^2 \int_0^{\frac{1}{3}\sqrt{4-x^2}} \int_{x^2+y^2}^{4-8y^2} dz\, dy\, dx.$$

$$V = 4 \int_0^2 \int_0^{\frac{1}{3}\sqrt{4-x^2}} (4 - x^2 - 9y^2)\, dy\, dx.$$

$$V = \frac{8}{9} \int_0^2 (4 - x^2)\sqrt{4 - x^2}\, dx = \frac{8}{3}\pi.$$

By similar reasoning, the volume of a solid whose bounding surfaces are expressed in cylindrical coordinates, may be formulated by a triple integral. This is done as follows:

Two sets of planes and one set of cylinders are constructed. One set of planes is taken perpendicular to the z-axis so that any two adjacent ones are separated by the distance Δz and one set is taken through the z-axis so that the angle between any two adjacent ones is equal to $\Delta \theta$. The set of right circular cylinders is taken with axes the z-axis so that any two consecutive ones have radii differing by the length $\Delta \rho$. These three sets of surfaces divide the solid into small elements. The base of each element is $\rho\, \Delta\rho\, \Delta\theta$ and its altitude is Δz. Hence, the volume of an element is

$$dV = \rho\, \Delta\rho\, \Delta\theta\, \Delta z.$$

The sum of the elements of volume which lie wholly within the solid is an approximation to the required volume. The limit of this sum is the volume of the solid, if the numbers of planes and cylinders become infinite. Thus,

$$V = \lim_{l,m,n \to \infty} \sum_{i=1}^{i=l} \sum_{j=1}^{j=m} \sum_{k=1}^{k=n} \rho_k \, \Delta\rho_k \, \Delta\theta_j \, \Delta z_i.$$

The notation denotes an element included between the kth and the $(k+1)$st cylinders, between the jth and the $(j+1)$st planes through the z-axis and between the ith and the $(i+1)$st planes perpendicular to the z-axis. The limit of this triple sum is expressed as a triple integral whose limits are obtained from the given boundaries of the solid.

To find the volume above the polar plane and included between the sphere and the cone,

$$\rho^2 + z^2 = 25, \quad 16z^2 - 9\rho^2 = 0,$$

we proceed as follows:

The element of volume is

$$dV = \rho \, \Delta z \, \Delta\rho \, \Delta\theta.$$

The z limits are from $\frac{3}{4}\rho$ to $\sqrt{25 - \rho^2}$, the ρ limits are from 0 to 4 and the θ limits are from 0 to 2π. Hence,

$$V = \int_0^{2\pi} \int_0^4 \int_{\frac{3}{4}\rho}^{\sqrt{25-\rho^2}} \rho \, dz \, d\rho \, d\theta$$

$$V = \int_0^{2\pi} \int_0^4 (\sqrt{25 - \rho^2} - \tfrac{3}{4}\rho) \, \rho \, d\rho \, d\theta = \frac{100\pi}{3}.$$

It is sometimes convenient to be able to formulate the volume of a solid by means of a triple integral in spherical coordinates. The element of volume in these coordinates is

$$dV = r^2 \sin \phi \, \Delta r \, \Delta\phi \, \Delta\theta,$$

which is derived by means of Figure 117 as follows:

A sphere TUV of radius r is taken with its center at the origin O. Through the point $P(r,\theta,\phi)$ the plane OVQ is taken through the OZ-axis and the plane SPR is taken perpendicular to the OZ-axis. In the first plane, QPV is the arc of a great circle of the sphere having the same radius as the sphere. In the second plane, SPW is the arc of a parallel which is a circle having the radius

$$RP = r \sin \phi.$$

If the angle ϕ is given the increment $\Delta\phi$, holding r and θ constant, the point P' is located on the arc QPV. Thus,

$$\angle VOP = \phi, \quad \angle POP' = \Delta\phi \quad \text{and} \quad PP' = r\,\Delta\phi.$$

If the angle θ is given the increment $\Delta\theta$, holding r and ϕ constant, the point P'' is located on the arc SPW. Thus,

$$\angle SRP = \theta, \quad \angle PRP'' = \Delta\theta \quad \text{and} \quad PP'' = r\sin\phi\,\Delta\theta.$$
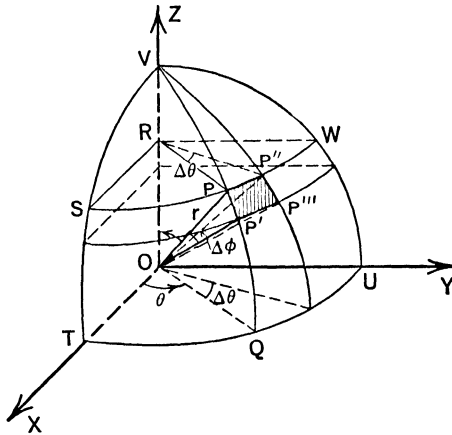
Fig. 117

Let the arc of the great circle VP'' and the parallel through P' intersect at the point P'''. The area of the enclosed surface $PP'P''P'''$ is approximately equal to

$$dS = PP' \cdot PP'' = r^2 \sin\phi\,\Delta\phi\,\Delta\theta,$$

for small increments of the angles.

A second sphere, not shown in the figure, is taken concentric with the first one and having a radius $r + \Delta r$. For a small value of Δr, there is formed a thin spherical shell between the two spheres. The small portion of this shell bounded by the planes POP', POP'', $P'OP'''$ and $P''OP'''$ is the *spherical element* of volume and is approximately equal to

$$dV = \Delta r\ dS,$$

for small values of the increments of the three variables. Replacing dS by the value found, we have the expression for the volume of an element in spherical coordinates given above.

The volume of a sphere $r = a$ is as follows:

$$V = 2\int_0^{2\pi} \int_0^{\pi/2} \int_0^a r^2 \sin \phi \, dr \, d\phi \, d\theta = \tfrac{4}{3}\pi a^3.$$

Using the element of surface area of a sphere obtained above, the area of the surface of a sphere of radius a is as follows:

$$S = 2\int_0^{2\pi} \int_0^{\pi/2} a^2 \sin \phi \, d\phi \, d\theta = 4\pi a^2.$$

Exercise 95

GROUP A

Evaluate each of the following integrals.

1. $\int_0^3 \int_0^{\sqrt{9-x^2}} \int_{x-y}^{x+y} z \, dz \, dy \, dx.$

2. $\int_0^4 \int_{1/y}^{e^y} \int_{1/z^2}^{y/z} z \, dx \, dz \, dy.$

3. $\int_2^4 \int_{1/z}^1 \int_0^{\sqrt{yz}} xyz \, dx \, dy \, dz.$

Find the volume of each of the following solids by triple integration.

4. Bounded by the coordinate planes and the plane $bcx + acy + abz = abc$.
5. Bounded by the surface $x^2 + y^2 = z + 1$ and $z = 0$.
6. The wedge cut from $x^2 + z^2 = a^2$ by $y = 0$ and $z = my$.
7. Bounded by $z = x^2 + 2y^2$, $x + y = 1$ and the coordinate planes.
8. Find the volume of a right circular cone, altitude h and radius of the base a, using triple integration and cylindrical coordinates.
9. Find the volume of a paraboloid of revolution, altitude and radius of the base are a, using triple integration and cylindrical coordinates.
10. Find the volume of a spherical zone cut from a sphere, radius $2a$, by one plane through the center and one plane parallel to it at a distance of a, using triple integration and spherical coordinates.

GROUP B

Find the volume of each of the following solids, using triple integration.

11. Enclosed by the surface $b^2c^2x^2 + a^2c^2y^2 + a^2b^2z^2 = a^2b^2c^2$.
12. Bounded by $z(x^2 + 4) = 8$ and the planes $x = y$, $x = 2$, $y = 0$ and $z = 0$.
13. Bounded by $\rho^2 + z^2 = 25$ and $4z = 3\rho$ above the $\rho\theta$-plane.
14. Bounded by $\rho^2 + z^2 = 23$ and the upper half of $z^2 = 9 + \rho^2$.
15. Bounded by $r^2 = 23$ and the upper half of $r^2 \cos 2\phi = 9$.

144. Moments.

The *moment of a force* about a line at right angles to the line of action of the force is defined as the *product of the force and the shortest distance*

between the two lines. The force is assumed to be exerted at a point P and the moment taken about a line L not through P. The moment of a force F taken about a line L as an axis is frequently said to be taken *with respect to the line L.* Qualitatively, the moment of F taken at P with respect to L is the *tendency* of a particle at P to turn around the axis L. If the force is the weight of the particle, the moment is the moment of the weight.
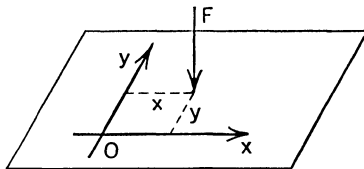
FIG. 118

In Figure 118, it is assumed that a force F is acting vertically to the horizontal xy-plane at the point $P(x,y)$. Then the moments of the force, with respect to the x- and the y-axes, are

$$F \cdot y \quad \text{and} \quad F \cdot x,$$

respectively. The signs of the moments depend on the signs of F, x and y.

Moment of Mass. The *moment of a mass m*, concentrated at a point P, with respect to a line L, is defined by the product

$$M = r \cdot m,$$

in which r is the shortest distance from P to L.

Let us assume a set of n particles of masses $m_1, m_2, m_3, \cdots, m_n$, concentrated at the points $(x_1,y_1), (x_2,y_2), (x_3,y_3), \cdots, (x_n,y_n)$, respectively, in the xy-plane. Then the moments of mass of the system with respect to the x-axis and with respect to the y-axis, are defined to be

$$M_x = y_1 m_1 + y_2 m_2 + y_3 m_3 + \cdots + y_n m_n = \sum_{i=1}^{i=n} y_i m_i$$

and

$$M_y = x_1 m_1 + x_2 m_2 + x_3 m_3 + \cdots + x_n m_n = \sum_{i=1}^{i=n} x_i m_i$$

respectively.

Moment of Inertia. The *moment of inertia of a mass m*, concentrated at a point P, with respect to a line L, is defined by the product

$$I = r^2 m,$$

in which r is the shortest distance from P to L. Since the square of the distance from the point to the axis is taken in this definition, the moment of inertia is sometimes called the *second moment of mass.*

Again, assume a set of n particles of masses m_i, concentrated at the corresponding points (x_i, y_i) in the xy-plane. Then the moments of inertia of the system with respect to the x-axis and with respect to the y-axis, are defined to be

$$I_x = \sum_{i=1}^{i=n} y_i{}^2 m_i \quad \text{and} \quad I_y = \sum_{i=1}^{i=n} x_i{}^2 m_i,$$

respectively.

The concepts and the definitions for moments of mass, thus far developed for masses concentrated at isolated points, admit of an extension to solids having continuous *homogeneous* mass distribution.

Assume an area S of a thin metal plate in the xy-plane. Corresponding to any point $P(x,y)$ within that area, an increment of area ΔS is chosen whose corresponding increment of mass is Δm. The ratio $\Delta m / \Delta S$ is the *average density* of the area ΔS. The limit of this ratio, as ΔS approaches zero, is the *density* k at the point P. Thus,

$$k = \lim_{\Delta S \to 0} \frac{\Delta m}{\Delta S} = \frac{dm}{dS}.$$

Hence,

$$dm = k \, dS = k \, \Delta x \, \Delta y.$$

It is to be observed that the thickness of the metal plate was disregarded in this discussion. The *element of mass* is the product of the element of area and the constant density factor.

Similarly, by means of an increment of volume ΔV whose increment of mass is Δm,

$$k = \lim_{\Delta V \to 0} \frac{\Delta m}{\Delta V} = \frac{dm}{dV}$$

and

$$dm = k \, dV = k \, \Delta x \, \Delta y \, \Delta z.$$

The moments of inertia for the thin metal plate are

$$I_x = \lim_{m,n \to \infty} \sum_{j=1}^{j=m} \sum_{i=1}^{i=n} k y_j{}^2 \, \Delta x_i \, \Delta y_j$$

and

$$I_y = \lim_{m,n \to \infty} \sum_{j=1}^{j=m} \sum_{i=1}^{i=n} k x_i{}^2 \, \Delta x_i \, \Delta y_j.$$

In Figure 119, the surface of a thin metal plate of uniform density is represented by the area OAC, bounded by the curve $y^2 = 4x$, the x-axis and the line $x = a$. The area is divided into elements Δx by Δy by equally spaced lines drawn parallel to the x-axis and to the y-axis. The element of mass is

$$dm = k\, \Delta x\, \Delta y.$$

Applying the fundamental theorem to the limits of the sums above, where the integrations are taken over the area of the plate, the moments of inertia are as follows:

$$I_x = k\int_0^a \int_0^{2\sqrt{x}} y^2\, dy\, dx = \frac{16}{15}\, ka^{5/2}.$$

$$I_y = k\int_0^a \int_0^{2\sqrt{x}} x^2\, dy\, dx = \frac{4}{7}\, ka^{7/2}.$$
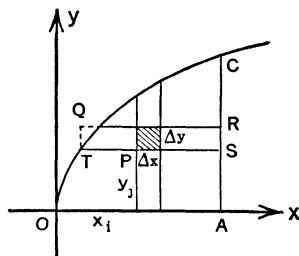
FIG. 119

The moment of inertia of a particle *about a point* is defined as the product of its mass and the square of its distance from the point. Consequently, the moment of inertia of a given plane area about the origin is

$$I_0 = k\int_a^b \int_{f_1(x)}^{f_2(x)} (x^2 + y^2)\, dy\, dx = I_x + I_y,$$

where the limits of integration are taken to cover the area of the surface.

The moment of inertia about the pole of a thin metal plate which is bounded by the polar axis and the upper half of the cardioid

$$\rho = 2(1 - \cos\theta)$$

is found as follows:

The elements of area and of mass are

$$dS = \rho\, \Delta\rho\, \Delta\theta, \quad dm = k\rho\, \Delta\rho\, \Delta\theta.$$

Hence,

$$I_0 = k\int_0^\pi \int_0^{2(1-\cos\theta)} \rho^3\, d\rho\, d\theta$$

$$= 4k\int_0^\pi (1 - \cos\theta)^4\, d\theta = \frac{35}{2}\, k\pi.$$

An extension of the above definitions enables one to find the moments of inertia of a solid about the coordinate axes and the origin. As the definitions for a plane area imply a double integration, the definitions for a solid imply a triple integration.

Consider the volume bounded by the paraboloid $z = \rho^2$ and the plane $z = 4$. The elements of volume and mass are

$$dV = \rho \, \Delta z \, \Delta \rho \, \Delta \theta, \quad dm = k\rho \, \Delta z \, \Delta \rho \, \Delta \theta.$$

For a point $P(\rho,\theta,z)$ at one vertex of the element of volume, the line segments PQ, PR and PS are drawn perpendicular to the axes OX, OY and OZ, respectively, as in Figure 120. Then, from the figure,

$$\overline{QP}^2 = z^2 + \rho^2 \sin^2 \theta, \quad \overline{RP}^2 = z^2 + \rho^2 \cos^2 \theta, \quad \overline{SP}^2 = \rho^2, \quad \overline{OP}^2 = z^2 + \rho^2.$$
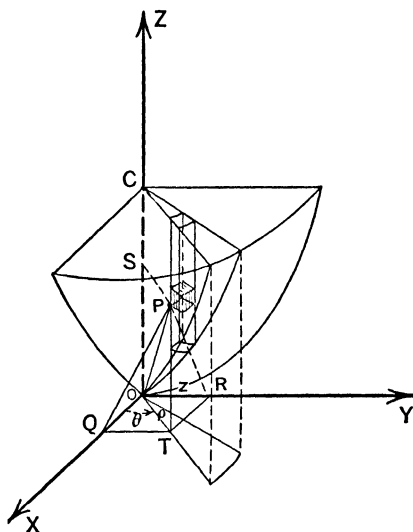
Hence the moments of inertia about OX, OY, OZ and O are

$$I_x = 4k \int_0^{\pi/2} \int_0^2 \int_{\rho^2}^4 \rho(z^2 + \rho^2 \sin^2 \theta) \, dz \, d\rho \, d\theta = \frac{1024}{15} k\pi,$$

$$I_y = 4k \int_0^{\pi/2} \int_0^2 \int_{\rho^2}^4 \rho(z^2 + \rho^2 \cos^2 \theta) \, dz \, d\rho \, d\theta,$$

$$I_z = 4k \int_0^{\pi/2} \int_0^2 \int_{\rho^2}^4 \rho^3 \, dz \, d\rho \, d\theta,$$

$$I_o = 4k \int_0^{\pi/2} \int_0^2 \int_{\rho^2}^4 \rho(z^2 + \rho^2) \, dz \, d\rho \, d\theta.$$

Because of symmetry relations, $I_x = I_y$.

Exercise 96

Find the following moments of inertia.

1. The area of a circle of radius a with respect to a diameter.
2. The area of a parabolic segment, base b and altitude a, with respect to its base.
3. The area of the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ with respect to its axes.
4. The area bounded by $xy = a^2$, $y = 0$, $x = a$ and $x = 2a$ with respect to the y-axis.
5. The area under one arch of $y = \sin x$ with respect to the x-axis.

Find the moment of inertia of each of the following solids with respect to the axes designated.

6. The solid bounded by $bcx + acy + abz = abc$ and the coordinate planes with respect to the z-axis.
7. The solid bounded by $x^3 = y^2$ and the planes $x = 1$, $z = 0$ and $z = 2$, with respect to the z-axis.
8. The solid bounded by $18z = 9x^2 + 4y^2$ and the plane $z = 2$ with respect to the z-axis.
9. The solid bounded by $4x^2 + 9y^2 = 36 - 18z$ and the plane $z = 0$ with respect to the x-axis.
10. The solid bounded by $\rho^2 + z^2 = a^2$, $\rho = a \cos \theta$ and the plane $z = 0$ with respect to the z-axis.
11. The solid bounded by $\rho = 2a \cos \theta$ and the planes $z = 0$ and $z = 2$ with respect to the z-axis.
12. The solid bounded by $\rho^2 + z^2 = 4$, $4z = 8 - \rho^2$ and the plane $z = 0$ with respect to the z-axis.

13. Find the moment of inertia with respect to the pole of the area inside $\rho = a(1 + \cos \theta)$ and outside $\rho = a$.
14. Find the moment of inertia of the area within $x = 2 \cos \theta$, $y = 3 \sin \theta$ with respect to the x-axis.

145. Centroid of a Plane Area.

As in the foregoing section, assume n particles having masses m_i concentrated at the corresponding points (x_i, y_i) in the xy-plane. Then, as before, the moments of mass are

$$M_x = \sum_{i=1}^{i=n} y_i m_i \quad \text{and} \quad M_y = \sum_{i=1}^{i=n} x_i m_i.$$

Suppose that a single particle has the mass $m = \sum_{i=1}^{i=n} m_i$ and that this mass is concentrated at a single point $\overline{P}(\overline{x}, \overline{y})$. Then its moments of mass with respect to the x-axis and with respect to the y-axis, are

$$M_x = m\overline{y} \quad \text{and} \quad M_y = m\overline{x}.$$

Hence,

$$m\bar{x} = \sum_{i=1}^{i=n} x_i m_i \quad \text{and} \quad m\bar{y} = \sum_{i=1}^{i=n} y_i m_i,$$

or

$$\bar{x} = \frac{x_1 m_1 + x_2 m_2 + \cdots + x_n m_n}{m_1 + m_2 + \cdots + m_n} = \frac{\displaystyle\sum_{i=1}^{i=n} x_i m_i}{\displaystyle\sum_{i=1}^{i=n} m} \cdot$$

and

$$\bar{y} = \frac{\displaystyle\sum_{i=1}^{i=n} y_i m_i}{\displaystyle\sum_{i=1}^{i=n} m_i} \cdot$$

The point $\overline{P}(\bar{x}, \bar{y})$ is known as the *center of gravity* or, the *center of mass* of the system of particles.
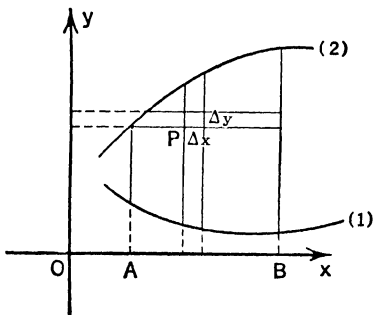
Fig. 121

The extension of these concepts to a solid having continuous homogeneous mass distribution enables us to find the coordinates of the center of gravity of such a solid. The moment of mass of the solid with respect to a coordinate axis is the limit of the sum of the moments of its elements of mass. The mass of the solid is the limit of the sum of its elements of mass.

Suppose that a thin · metal plate whose area is S in Figure 121, in the xy-plane is bounded by the curves

$$y = f_1(x) \quad \text{and} \quad y = f_2(x)$$

and the lines $x = a$ and $x = b$. Then the coordinates of the center of gravity of the area S are expressed as follows:

$$\bar{x} = \frac{\displaystyle\lim_{m,n\to\infty} \sum_{j=1}^{j=m} \sum_{i=1}^{i=n} k x_i \Delta x_i \Delta y_j}{\displaystyle\lim_{m,n\to\infty} \sum_{j=1}^{j=m} \sum_{i=1}^{i=n} k \Delta x_i \Delta y_j} = \frac{\displaystyle\int_a^b \int_{f_1(x)}^{f_2(x)} x \, dy \, dx}{\displaystyle\int_a^b \int_{f_1(x)}^{f_2(x)} dy \, dx} \cdot$$

$$\bar{y} = \frac{\displaystyle\int_a^b \int_{f_1(x)}^{f_2(x)} y \, dy \, dx}{S} \cdot$$

In this study we are concerned with the coordinates $(\bar{x}, \bar{y})$ for a thin metal plate whose surface area is S and, being a homogeneous solid of uni-

form thickness, it has a constant density k. The density k is a factor of both numerator and denominator of the expressions written for $\bar{x}$ and $\bar{y}$. Consequently, the quotient of the limit of the sum of the mass-moments, with respect to an axis, and the limit of the sum of the mass elements, is independent of k and, therefore, *independent of gravity*. In other words, our study is reduced to that of geometrical figures. It is for this reason that it is customary to speak of the point $P(\bar{x},\bar{y})$ as the *centroid* of the figure rather than its center of gravity.

Let us discuss more specifically the difference between the center of gravity of a thin solid and the centroid of its surface. The centroid of a plane area such as S in Figure 121 is a point $\bar{P}$ which is located by its coordinates $\bar{x}$ and $\bar{y}$ with reference to a coordinate system. This in turn, locates the point $\bar{P}$ with reference to the area S itself and may be taken without reference to any coordinate system. Suppose, having located the centroid $\bar{P}$ with reference to S, we take a thin metal plate of uniform thickness and density so that the upper and lower surfaces are exactly the size and shape of S. If this plate is placed in a horizontal position and if it is supported by a needle point, placed at a point Q on the under surface vertically below $\bar{P}$ on the surface S, the *plate will balance*. The center of gravity of the solid plate will be at a point midway between Q and the centroid $\bar{P}$ of the area S.
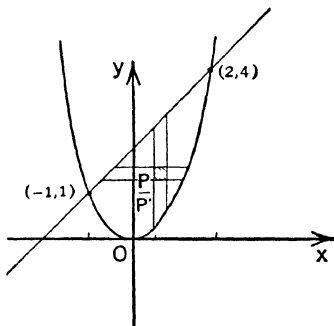
Fɪɢ. 122

Let us find the coordinates $(\bar{x},\bar{y})$ of the centroid of a thin plate of uniform density which is bounded by the parabola $y = x^2$ and the line $y = x + 2$.

The point $P(x,y)$ is chosen at a vertex of an element of area Δx by Δy as in Figure 122. Using the equations derived for the coordinates of $\bar{P}$, we have

$$\bar{x} = \frac{\int_{-1}^{2}\int_{x^2}^{x+2} x\,dy\,dx}{\int_{-1}^{2}\int_{x^2}^{x+2} dy\,dx} = \frac{\int_{-1}^{2}(x^2 + 2x - x^3)\,dx}{\int_{-1}^{2}(x + 2 - x^2)\,dx} = \frac{\dfrac{9}{4}}{\dfrac{9}{2}} = \frac{1}{2},$$

$$\bar{y} = \frac{\int_{-1}^{2}\int_{x^2}^{x+2} y\,dy\,dx}{S} = \frac{\dfrac{1}{2}\int_{-1}^{2}(x^2 + 4x + 4 - x^4)\,dx}{S} = \frac{\dfrac{36}{5}}{\dfrac{9}{2}} = \frac{8}{5}.$$

If an area has boundaries which are expressed in the polar coordinate system, where the polar axis is the x-axis, the coordinates of the centroid may be written immediately:

$$\bar{x} = \bar{\rho} \cos \bar{\theta} = \frac{\displaystyle\int_{\alpha}^{\beta} \int_{f_1(\theta)}^{f_2(\theta)} \rho^2 \cos \theta \, d\rho \, d\theta}{\displaystyle\int_{\alpha}^{\beta} \int_{f_1(\theta)}^{f_2(\theta)} \rho \, d\rho \, d\theta},$$

$$\bar{y} = \bar{\rho} \sin \bar{\theta} = \frac{\displaystyle\int_{\alpha}^{\beta} \int_{f_1(\theta)}^{f_2(\theta)} \rho^2 \sin \theta \, d\rho \, d\theta}{\displaystyle\int_{\alpha}^{\beta} \int_{f_1(\theta)}^{f_2(\theta)} \rho \, d\rho \, d\theta}.$$

Exercise 97

GROUP A

Find the coordinates of the centroid of each of the following areas.

1. A right triangle having a base b and altitude a.
2. A semicircular area for a circle of radius a.
3. One quadrant of the ellipse $b^2x^2 + a^2y^2 = a^2b^2$.
4. The area bounded by the parabolas $y^2 = 4x$ and $x^2 = 4y$.
5. The area bounded by $x^{1/2} + y^{1/2} = a^{1/2}$ and the coordinate axes.
6. The area in the second quadrant under the curve $y = c^x$.
7. The area inside the loop of the curve $y^2 = 4x^2 - x^3$.
8. The area inside the curve $\rho = a(1 + \cos \theta)$.

GROUP B

Find the coordinates of the centroid of each of the following areas.

9. One arch of the cycloid $x = a(\theta - \sin \theta)$, $y = a(1 - \cos \theta)$.
10. The area in the first quadrant between $b^2x^2 + a^2y^2 = a^2b^2$ and $x^2 + y^2 = a^2$, where $a > b$.
11. The area between $x^2 + y^2 = a^2$ and the tangents $x = a$ and $y = a$ in the first quadrant.
12. The area bounded by $y^2(2a - x) = x^3$ and the line $x = 2a$.
13. The area under $y = \sin x$ from $x = 0$ to $2x = \pi$.
14. The area in the first quadrant bounded by $y(x^2 + 4) = 8$, $x^2 = 4y$ and the y-axis.
15. The area of one loop of the curve $\rho = a \cos 2\theta$.
16. The area of one loop of the curve $\rho = a \cos 3\theta$.

146. Centroid of an Arc of a Plane Curve.

The coordinates of the centroid of an arc of a plane curve may be found by considering the moments of an element of the arc about the coordinate

axes. As in the previous sections,

$$\bar{x} = \frac{\int_C x\, ds}{\int_C ds}, \qquad \bar{y} = \frac{\int_C y\, ds}{\int_C ds},$$

where the integrations are taken along the curve.

The coordinates of the centroid of the first quadrant arc of a circle of radius a, by rectangular coordinates are found as follows:

Since

$$x^2 + y^2 = a^2 \quad \text{and} \quad ds = \frac{a}{y}\, dx,$$

$$\bar{x} = \frac{a\int_0^a \frac{x}{y}\, dx}{a\int_0^a \frac{1}{y}\, dx} = \frac{2}{\pi} a.$$

The ordinate of the centroid is equal to $\bar{x}$ because of the symmetry relations.

If the coordinates are to be found by means of the parametric representation of the circle,

$$x = a \cos \theta, \quad y = a \sin \theta \quad \text{and} \quad ds = a\, d\theta.$$

Hence,

$$\bar{y} = \frac{a^2 \int_0^{\pi/2} \sin \theta\, d\theta}{a \int_0^{\pi/2} d\theta} = \frac{2}{\pi} a.$$

Exercise 98

Find the coordinates of the centroid of each of the following arcs.

1. The semi-circular arc of $x^2 + y^2 = a^2$ in the first and second quadrants.
2. A circular arc of $x^2 + y^2 = a^2$ which subtends an angle 2α which is bisected by the x-axis.
3. The arc of the curve $y = \frac{a}{2}(e^{x/a} + e^{-x/a})$ from $x = -a$ to $x = a$.
4. The arc of the first arch of $x = a(\theta - \sin \theta)$, $y = a(1 - \cos \theta)$.
5. The first quadrant arc of $x^{2/3} + y^{2/3} = a^{2/3}$.

147. Centroid of a Volume.

The moment of mass of an element of mass about a *plane* is the product of the mass and the distance from the plane. Pursuing the same type of reasoning as was given in finding the coordinates of the centroid of an area,

the moments of mass of a system of n particles with respect to the coordinate planes xy, xz and yz are

$$M_{xy} = \sum_{i=1}^{i=n} z_i m_i, \quad M_{xz} = \sum_{i=1}^{i=n} y_i m_i, \quad \text{and} \quad M_{yz} = \sum_{i=1}^{i=n} x_i m_i,$$

respectively. As in Section 144, assume that the mass of the system is concentrated at the centroid $\overline{P}(\overline{x}, \overline{y}, \overline{z})$ and also, that the same reasoning is carried out to permit application to a continuous homogeneous mass occupying a volume V. Then we have

$$M_{yz} = kV\overline{x}, \quad M_{xz} = kV\overline{y}, \quad M_{xy} = kV\overline{z}.$$

Hence,

$$\overline{x} = \frac{M_{yz}}{kV} = \frac{\displaystyle\int_a^b \int_{y_1}^{y_2} \int_{z_1}^{z_2} x \, dz \, dy \, dx}{\displaystyle\int_a^b \int_{y_1}^{y_2} \int_{z_1}^{z_2} dz \, dy \, dx},$$

$$\overline{y} = \frac{\displaystyle\int_a^b \int_{y_1}^{y_2} \int_{z_1}^{z_2} y \, dz \, dy \, dx}{V}, \quad \overline{z} = \frac{\displaystyle\int_a^b \int_{y_1}^{y_2} \int_{z_1}^{z_2} z \, dz \, dy \, dx}{V}.$$

Exercise 99

Find coordinates of the centroid of each of the following solids.

1. The solid bounded by $bcx + acy + abz = abc$ and the coordinate planes.
2. One eighth of a sphere of radius a.
3. The solid between the coordinate planes and the first octant portion of the surface $b^2c^2x^2 + a^2c^2y^2 + a^2b^2z^2 = a^2b^2c^2$.
4. The solid in the first octant bounded by $x^2 + z^2 = a^2$ and $x^2 + y^2 = a^2$.
5. The solid bounded by the surface $18z = 4x^2 + 9y^2$ and the planes $x = 0$, $y = 0$ and $z = 2$.
6. The solid in the first octant bounded by $az = \rho^2$ and $z + \rho = 2a$.
7. The solid in the first octant bounded by $z = \rho$ and $z^2 + \rho^2 = 1$.
8. The solid above the $\rho\theta$-plane bounded by $\rho^2 + z^2 = a^2$ and $\rho = b$ where $a > b$.

148. Attraction.

Two particles of matter attract each other with a force which is directly proportional to the product of their masses and inversely proportional to the square of the distance between them. The force is called the *attraction* of one particle on the other. This statement is also known as the *gravitational law*.

If the masses of two particles are m_1 and m_2 and the distance between them is r, the force F is

$$F = \lambda \frac{m_1 m_2}{r^2},$$

where λ is the constant of proportionality.

Consider n particles in the xy-plane at the points $P_1, P_2, P_3, \cdots P_n$ having the masses $m_1, m_2, m_3, \cdots m_n$, respectively, as shown in Figure 123.
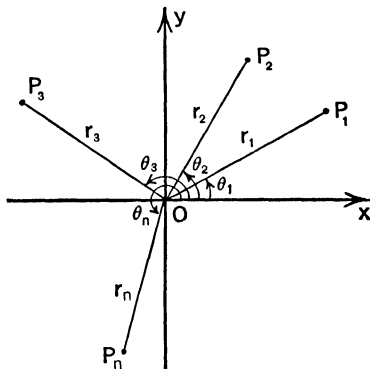
FIG. 123

Let the distances of the particles from the origin be denoted by $r_1, r_2, r_3, \cdots r_n$.

If it desired to find the attraction of the sum of these masses on a *unit* particle at the origin, the attractions of the individual particles are

$$F_1 = \lambda \frac{m_1}{r_1{}^2}, \quad F_2 = \lambda \frac{m_2}{r_2{}^2}, \quad F_3 = \lambda \frac{m_3}{r_3{}^2}, \cdots F_n = \lambda \frac{m_n}{r_n{}^2}.$$

But since these forces are not parallel, they cannot be added directly. To find their resultant, each may be resolved into its x and y-components. Assuming that

$$\angle xOP_1 = \theta_1, \quad \angle xOP_2 = \theta_2, \quad \angle xOP_3 = \theta_3, \cdots \angle xOP_n = \theta_n,$$

the x- and the y-components of the resultant attraction are

$$F_x = \lambda \left(\frac{m_1}{r_1{}^2} \cos \theta_1 + \frac{m_2}{r_2{}^2} \cos \theta_2 + \frac{m_3}{r_3{}^2} \cos \theta_3 + \cdots + \frac{m_n}{r_n{}^2} \cos \theta_n \right),$$

$$= \lambda \sum_{i=1}^{i=n} \frac{m_i}{r_i{}^2} \cos \theta_i,$$

$$F_y = \lambda \left(\frac{m_1}{r_1{}^2} \sin \theta_1 + \frac{m_2}{r_2{}^2} \sin \theta_2 + \frac{m_3}{r_3{}^2} \sin \theta_3 + \cdots + \frac{m_n}{r_n{}^2} \sin \theta_n \right),$$

$$= \lambda \sum_{i=1}^{i=n} \frac{m_i}{r_i{}^2} \sin \theta_i.$$

The resultant of the attraction is equal to

$$F = \sqrt{F_x{}^2 + F_y{}^2}$$

and acts in the direction with the x-axis equal to

$$\theta = \arctan \frac{F_y}{F_x}.$$

The finding of the attraction of isolated particles in a plane on a unit mass is preliminary to the finding of the attraction of a solid of uniform density on a unit mass.

Let the solid be divided into n elements, the mass of each of which is approximately dm. Also let $P(x,y)$ be a point at which the mass of any one element is considered to be concentrated then the attraction of this element on a unit mass at the origin is approximately

$$dF = \lambda \frac{dm}{r^2}$$

and its x- and y-components are

$$dF_x = \lambda \frac{dm}{r^2} \cos\theta, \quad dF_y = \lambda \frac{dm}{r^2} \sin\theta,$$

where $r = OP$ and $\theta = \angle xOP$. Hence, the sums of the x- and the y-components of the attraction of all the elements on the unit mass at the origin are

$$F_x = \lambda \int_M \frac{\cos\theta}{r^2}\, dm, \quad F_y = \lambda \int_M \frac{\sin\theta}{r^2}\, dm,$$

where the integrations are taken over the mass in question.
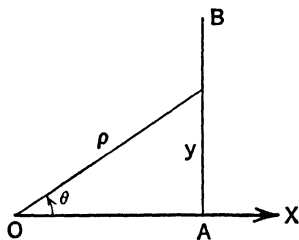
FIG. 124

To find the attraction due to a slender straight wire of length b on a unit mass which lies on a line perpendicular to the wire at one extremity and a units from it, we proceed as follows:

Let the wire AB in Figure 124 lie on the $\rho\theta$-plane with one end on the polar axis and let the unit mass lie at the pole. If an xy-coordinate system is used with the x-axis the polar axis, the coordinates of any point P are (x,y) or (ρ,θ). Also,

$$\rho^2 = x^2 + y^2, \quad \rho \cos\theta = a, \quad \rho \sin\theta = y.$$

If the density of any element Δy is the constant k,

$$dm = k \, \Delta y$$

and

$$F_x = \lambda k \int_0^b \frac{\cos \theta}{\rho^2} \, dy, \quad F_y = \lambda k \int_0^b \frac{\sin \theta}{\rho^2} \, dy.$$

Expressing the limits and the variable of integration in polar coordinates, and integrating, we have

$$F_x = \frac{\lambda}{a} k \int_0^\alpha \cos \theta \, d\theta = \frac{\lambda}{a} k \sin \alpha,$$

$$F_y = \frac{\lambda}{a} k \int_0^\alpha \sin \theta \, d\theta = \frac{\lambda}{a} k(1 - \cos \alpha),$$

where $y = a \tan \theta$, $dy = a \sec^2 \theta \, d\theta$ and $\alpha = \arctan \dfrac{b}{a}$. Hence, the resultant force of the attraction is

$$F = \frac{\lambda}{a} k \sqrt{2 - 2 \cos \alpha} = \frac{2}{a} \lambda k \sin \frac{\alpha}{2},$$

and the direction of that force is

$$\beta = \arctan \left[\frac{1 - \cos \alpha}{\sin \alpha} \right] = \frac{\alpha}{2}.$$
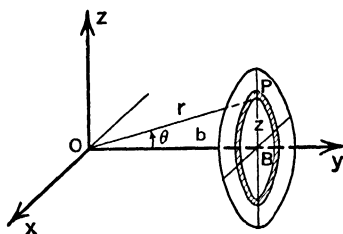
Fɪɢ. 125

To find the attraction due to a thin circular homogeneous metal plate of radius a on a unit mass which lies on a line perpendicular to the plane of the plate at its center and b units from it, we proceed as follows:

The plate is taken as in Figure 125, perpendicular to the y-axis with its center at the point $B(0,b,0)$. At any point $P(0,b,z)$ on the plate a ring-

shaped element is taken whose mass is approximately,

$$dm = 2\pi kz \, dz.$$

Then,

$$F_x = 0, \quad F_z = 0,$$

and

$$F_y = 2\pi k\lambda \int_0^a \frac{\cos \theta}{r^2} z \, dz$$

$$F_y = 2\pi k\lambda b \int_0^a \frac{z \, dz}{(z^2 - b^2)^{3/2}} = 2\pi k\lambda \frac{\sqrt{a^2 + b^2} - b}{\sqrt{a^2 + b^2}},$$

where $r \cos \theta = b$ and $r^2 = z^2 + b^2$.

To find the attraction due to a right circular cylindrical homogeneous solid of radius a and length l on a unit mass which lies on the axis of the cylinder at a distance of b units from the nearer base, we proceed as follows:

The cylinder is taken as in Figure 126, with its axis on the y-axis and the centers of its bases at the points $B_1(0,b,0)$ and $B_2(0,b + l,0)$. At any
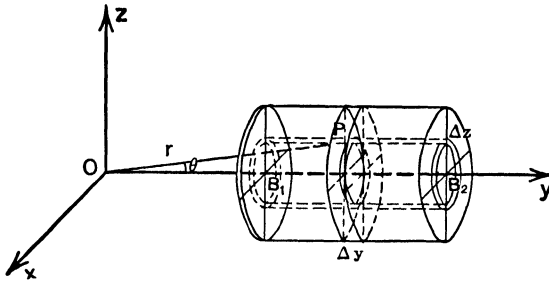
Fig. 126

point $P(0,y,z)$ within the cylinder a circular ring-shaped element is taken, whose mass is approximately

$$dm = 2\pi kz \, \Delta z \, \Delta y.$$

Hence

$$F_y = 2\pi k\lambda \int_b^{b+l} \int_0^a \frac{\cos \theta}{r^2} z \, dz \, dy$$

$$= 2\pi k\lambda [l + \sqrt{a^2 + b^2} - \sqrt{a^2 + (b + l)^2}].$$

where $y = r \cos \theta$ and $r^2 = y^2 + z^2$.

Exercise 100

Find the attraction of each of the following on a unit mass located as indicated.

1. A quadrant arc of a circular slender wire of radius a, unit mass at the center.

2. A slender straight wire of length $2a$, unit mass at a distance of b on a line perpendicular to the center of the wire.

3. A hemispherical shell of radius a, unit mass at the center.

4. A right circular cylindrical shell, radius of the base a and altitude h, unit mass at the center of the base.

5. A hemispherical solid of radius a, unit mass at a distance of a from the base on a line perpendicular to the base at its center and away from the solid.

6. A right circular conical solid, radius of the base 2 and altitude 2, unit mass at the vertex of the cone.

7. A solid in the shape of the volume generated by the revolution of the area bounded by the curve $x^2 = y^3$ and the line $y = 3$ about the y-axis, unit mass at the origin.

8. A solid sphere of radius a, unit mass at a point on the surface of the sphere.

CHAPTER XVII

149. Differential Equations.

A *differential equation* is an equation which involves derivatives or differentials.

The subject known as differential equations is an extension of the calculus rather than a part of it. ˈAny adequate treatment of this important subject is far beyond the scope of this book. However, it is advisable that the student of the elementary calculus be familiar with some few types of differential equations which arise in the physical sciences.

Those differential equations in which all the derivatives have reference to a single independent variable are called *ordinary* differential equations as opposed to *partial* differential equations in which there are two or more independent variables and partial derivatives with reference to one or more of them.

The *order* of a differential equation is the order of the highest derivative appearing in the equation.

The *degree* of a differential equation is the degree of the highest derivative when the equation is free of radicals and fractions.

The differential equation

$$\frac{dy}{dx} = y, \quad dy = y \, dx,$$

is of the first order and first degree.

The differential equation

$$\frac{d^2y}{dx^2} = 6x,$$

is of the second order and first degree.

The differential equation

$$\left(\frac{dy}{dx}\right)^2 = a^2x^3,$$

is of the first order and second degree.

The study which is made in this chapter is limited to those differential

equations which are of no higher order and degree than the second. More-over, the treatment of even these simpler types of equations is by no means a complete one.

150. Solutions of Differential Equations.

The *solution* or the *integral* of a differential equation is a relation between the variables, not containing their derivatives, by means of which the differential equation is satisfied.

The solution of the first equation given in the preceding section is

$$\int \frac{dy}{y} = \int dx + C,$$

$$\ln y = x + C, \quad y = Ae^x.$$

The solution of the second, is

$$\frac{dy}{dx} = 6 \int x \, dx + C_1 = 3x^2 + C_1,$$

$$y = 3 \int x^2 \, dx + C_1 \int dx + C_2,$$

$$y = x^3 + C_1 x + C_2.$$

The solution of the third, is

$$y = a \int x^{3/2} \, dx + C$$

$$5y = 2ax^{5/2} + C', \quad 25(y + B)^2 = 4a^2x^5.$$

The arbitrary constants A, C_1, C_2 and B appearing in these solutions are called the *arbitrary constants of integration*. In general, the solution of a differential equation which contains a number of arbitrary constants equal to the order of the equation is called the *general solution* or *complete integral*. Solutions which are obtained from it, by giving particular values to the constants, are called *particular solutions*.

In the last solution obtained above, the result is a complete integral in which B is the only arbitrary constant of integration, since the equation is of the first order. The general solution of the equation

$$x \, dx + y \, dy = 0,$$

is

$$x^2 + y^2 = a^2,$$

in which a^2 is the arbitrary constant. Assuming values for a, the particular solutions may be obtained, $x^2 + y^2 = 1$, $x^2 + y^2 = 3$, $x^2 + y^2 = 16$, etc.

151. Geometric Interpretation of Differential Equations.

Consider a differential equation of the first degree and first order represented by the symbol

$$f\left(x, y, \frac{dy}{dx}\right) = 0.$$

Let the general solution be denoted by the symbol

$$F(x, y, C) = 0.$$

For each particular value of C, the resulting equation is interpreted as a curve in the xy-plane. Hence, for all values of C, the locus of the latter equation is a *system of plane curves*. The same interpretation is obtained by an analysis of the first equation, which may be done as follows:

A point $P(x, y)$ which moves, subject to the conditions of the first equation, must have the direction at any one of its positions, (x_1, y_1), given by the corresponding value obtained from dy/dx for $x = x_1$ and $y = y_1$. Thus, the point P will describe a curve, the coordinates of every point of which, and the direction of the tangent thereat, will satisfy the differential equation. If the moving point P starts at some other point of the plane not on the curve already described, and if it moves in the same manner, it will trace another curve, the coordinates of whose points and the directions of the tangents thereat, satisfy the same differential equation. Continuing in this way, through every point of the plane, there will pass a particular curve, for every point of which

$$x, y, \frac{dy}{dx},$$

will satisfy the given differential equation.

The *locus of a first degree and first order differential equation is a singly infinite system of plane curves*, each curve of which is the locus of a particular solution of that equation. That the system of curves is *singly infinite* is equivalent to saying that every ordinary point of the xy-plane lies on *one*, and *only one*, curve of the system.

The locus of the equation

$$x \, dx - y \, dy = 0$$

is a singly infinite system of hyperbolas. This may be seen from the general solution as follows:

From the solution

$$x^2 - y^2 = C,$$

if $C = 0, \quad x - y = 0 \quad \text{and} \quad x + y = 0.$

This curve of the system is a pair of lines bisecting the first and the second quadrants as shown in Figure 127.

From the same solution, if $C > 0$, let $C = a^2$. The solution is then written

$$x^2 - y^2 = a^2.$$

For all values of a, other than zero, this is the equation of all those hyperbolas asymptotic to the given pair of lines, symmetrical with respect to the coordinate axes with transverse axes on the x-axis. Three curves, A_1, A_2 and A_3, of the system are drawn in the figure.
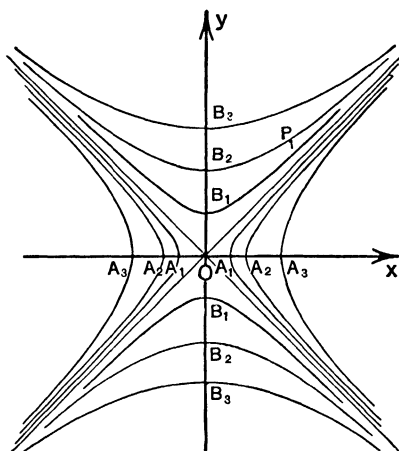
FIG. 127

Again, from the same solution, if $C < 0$, let $C = -b^2$. The solution is then written

$$y^2 - x^2 = b^2.$$

For all values of b, other than zero, this is the equation of all those hyperbolas asymptotic to the given pair of lines, symmetrical with respect to the coordinate axes with transverse axes on the y-axis. Three curves, B_1, B_2, and B_3, of the system are drawn in the figure.

These three forms of the solution of the given equation constitute the equations of the curves of the system. Through any one point, except the origin, passes but one curve of the system. However, in this illustration, $dy/dx = x/y$ is not defined at the origin. For example, through the point

$P_1(3,4)$ passes the curve $y^2 - x^2 = 7$, one hyperbola of the last mentioned set.

Let us consider the geometric interpretation of a differential equation of the first degree and second order. Such an equation may be represented by the symbol

$$f\left(x, y, \frac{d^2y}{dx^2}\right) = 0,$$

in which it is assumed, for simplicity, that no term in dy/dx is present in the equation. Let the general solution be denoted by the symbol

$$F(x, y, C_1, C_2) = 0,$$

since such an equation is obtained by two successive integrations of the differential equation.

Corresponding to each particular value of C_1, is an equation containing the arbitrary constant C_2. Hence, in general, for every different value assigned to the constant C_1, the equation containing the constant C_2 is interpreted as a singly infinite system of curves. This leads to the statement that *the locus corresponding to a first degree and second order differential equation is a doubly infinite system of curves.* That the system of curves is *doubly infinite* is equivalent to saying that an infinity of curves pass through each point of the xy-plane.

The solution of the differential equation

$$\frac{d^2y}{dx^2} = 2 \quad \text{is} \quad y = x^2 + C_1 x + C_2.$$

If

$$C_1 = 2, \quad (x + 1)^2 = y - C_2 + 1.$$

This is the equation of a system of parabolas all of whose axes are the line $x + 1 = 0$. Such a singly infinite system is obtained for each particular choice of a value for C_1. Since any number of choices for C_1 are possible, there is a singly infinite system of such systems. In this instance, all the curves are parabolas, with certain degenerate cases, each having its axis parallel to the y-axis. Through any point pass any number of parabolas. For example, if the point is $P_1(1,1)$, from the solution the only condition to be satisfied is that $C_1 + C_2 = 0$. Since there are an unlimited number of choices possible, satisfying this condition, there are an unlimited number of curves of the system passing through P_1.

In conclusion, it may also be said that in general, *two* curves of the system of curves corresponding to a differential equation of the first order and second degree pass through each point of the plane.

Exercise 101

GROUP A

1. Solve the equation $x + y \dfrac{dy}{dx} = 0$ and by means of the result give the geometric interpretation.

2. Find the general solution of $4x + y \dfrac{dy}{dx} = 0$ and find the particular solution which is satisfied by the point (2,3).

3. If $f\left(x, y, \dfrac{dy}{dx}\right) = 0$ represents the differential equation in Problem 2, show that $(2,3, -\tfrac{8}{3})$ satisfies that equation.

4. Show that the general solution of the equation $(x - 2) + (y + 2)\dfrac{dy}{dx} = 0$ is the system of circles $(x - 2)^2 + (y + 2)^2 = a^2$.

5. Find the differential equation for which the general solution is $x^2 - y^2 + y^3 = C$.

6. Find the general solution of the equation $\dfrac{d^2 y}{dx^2} = 6x + 2$. Find the equation of the system of curves through the point $A(1,2)$. Find the equation of the curve through the points A and $B(2,3)$.

7. Show that $x^2 + 4y = 0$ is a solution of $\left(\dfrac{dy}{dx}\right)^2 + x\dfrac{dy}{dx} - y = 0$.

8. Show that $y = 2x^2 + 3x$ is a solution of $x^2 \dfrac{d^2 y}{dx^2} - 2x\dfrac{dy}{dx} + 2y = 0$.

GROUP B

Solve each of the following equations and show that the solution satisfies the given equation.

9. $(y + 1) + (x + 2)\dfrac{dy}{dx} = 0.$

12. $\left(\dfrac{dy}{dx}\right)^2 = x^2(x^2 + 4).$

10. $\dfrac{dy}{dx} + x = e^x.$

13. $\left(\dfrac{dy}{dx}\right)^2 + 2\left(\dfrac{dy}{dx}\right) = 4x^2 - 1.$

11. $\dfrac{1}{y}\dfrac{dy}{dx} + 1 = \cos x.$

14. $x\dfrac{d^2 y}{dx^2} = 2.$

15. Eliminate the constant C from the equation $y = Cx$ by forming the differential equation. Solve the equation found and show that its solution is the given equation.

16. Eliminate the constants from the equation $y = Ax + Bx^2$.

152. Separation of Variables.

Every differential equation of the first order and first degree can be written in the form

$$f_1(x,y)\ dx + f_2(x,y)\ dy = 0.$$

It is often possible to transform the equation so that the coefficients of dx and dy are functions of x and y alone, respectively. This transformation to the form,

$$g(x) \, dx + q(y) \, dy = 0,$$

is known as *separation of variables*.

A differential equation in which the variables are separable, is solved by direct integration. For example, the equation

$$y - x \frac{dy}{dx} = y^2 + \frac{dy}{dx}$$

transforms to

$$(y - y^2) \, dx - (x + 1) \, dy = 0,$$

$$\frac{dx}{x + 1} - \frac{dy}{y - y^2} = 0.$$

Integrating,

$$\ln \frac{(x + 1)(1 - y)}{y} = C,$$

$$(1 + x)(1 - y) = Ay.$$

It is possible to write the solution of a differential equation in a variety of forms. The differences in such forms are due to the treatment of the arbitrary constant. For example, in the illustration above,

$$A = e^C.$$

Usually, preference is shown for that form which gives the simplest expression of the solution.

Exercise 102

GROUP A

Solve each of the following equations.

1. $xy + 2 \dfrac{dy}{dx} = 0.$

2. $e^y \sin x + \dfrac{dy}{dx} = 0.$

3. $(1 + y^2) + y(1 + x^2) \dfrac{dy}{dx} = 0.$

4. $(1 + x)y^2 \, dx - x^3 \, dy = 0.$

5. $2(1 - y^2)xy \, dx + (1 + x^2)(1 + y^2) \, dy = 0.$

6. $\sin x \cos^2 y \, dx + \cos^2 x \, dy = 0.$

7. Find the differential equation of all circles having their centers at the origin.   Verify the equation by solving it.

8. Find the differential equation of all parabolas having their vertices at the origin symmetrical to the x-axis.   Verify the equation by solving it.

9. Find the differential equation of all parabolas having their vertices at the origin symmetrical to the y-axis.   Verify the equation by solving it.

10. Find the differential equation of all circles of radius 2 having their centers on the x-axis.   Verify the equation by solving it.

GROUP B

Find the differential equation for each of the following systems of curves.

11. All circles through the origin having their centers on the x-axis.

12. All circles through the origin having their centers on the y-axis.

13. All parabolas having their foci at the origin and symmetrical with respect to the x-axis.

14. All straight lines whose x- and y-intercepts have the sum of 4.

15. All straight lines tangent to the circle $x^2 + y^2 = 9$.

16. All straight lines tangent to the hyperbola $x^2 - y^2 = 4$.

153.   Equations Homogeneous in x and y.

A polynomial in x and y is said to be *homogeneous* when all the terms are of the same degree in x and y.   In general, any function of x and y is a homogeneous function of the nth degree if, when x and y are replaced by kx and ky, the result is the original function multiplied by k^n.   For example, the functions

$$x^2 - xy + y^2, \quad 2x^3 - 3x^2y + 4xy^2$$

are homogeneous functions of the second and third degrees, respectively. Also, the function

$$2x - y + xe^{y/x}$$

is a homogeneous first degree function.

A differential equation of the first order and degree

$$f_1(x,y) \, dx + f_2(x,y) \, dy = 0,$$

may be written in the form

$$\frac{dy}{dx} + \frac{f_1(x,y)}{f_2(x,y)} = 0.$$

If the functions f_1 and f_2 are homogeneous of the same degree in x and y, it is *always* possible to effect a separation of variables by the transformation

$$y = vx.$$

From it,

$$\frac{dy}{dx} = v + x\frac{dv}{dx}.$$

The equation is transformed to

$$v + x\frac{dv}{dx} + F(v) = 0,$$

where

$$\frac{f_1(x,y)}{f_2(x,y)} = \frac{f_1(x,vx)}{f_2(x,vx)} = F(v) \quad \text{for} \quad y = vx.$$

For example, if

$$f_1(x,y) = x^2 + xy - y^2 \quad \text{and} \quad f_2(x,y) = x^2 - y^2,$$

for $y = vx$,
$$\frac{f_1(x,vx)}{f_2(x,vx)} = \frac{x^2 + vx^2 - v^2x^2}{x^2 - v^2x^2}$$

$$= \frac{1 + v - v^2}{1 - v^2} = F(v).$$

The separation of variables gives the differential equation

$$\frac{dx}{x} + \frac{dv}{F(v) + v} = 0.$$

When this equation is solved by the methods of the preceding section and when the reverse substitution

$$v = \frac{y}{x},$$

is made, the solution of the given differential equation has been found.

The solution of the differential equation

$$x + y + x\frac{dy}{dx} = 0$$

is obtained as follows:

$$\frac{dy}{dx} + \frac{x + y}{x} = 0.$$

Letting $y = vx$,

$$v + x\frac{dv}{dx} + (1 + v) = 0,$$

$$\frac{dx}{x} + \frac{dv}{2v + 1} = 0.$$

Integrating,

$$\ln x\sqrt{2v + 1} = C,$$
$$x^2(2v + 1) = A.$$

Substituting

$$v = \frac{y}{x},$$

$$x^2 + 2xy = A.$$

As a second illustration of the method of solving differential equations of this type, the equation

$$x - y + xe^{y/x} + x\frac{dy}{dx} = 0,$$

is solved as follows:
Letting $y = vx$,

$$(1 + e^v) + x\frac{dv}{dx} = 0.$$

Separating variables,

$$\frac{dx}{x} + \frac{dv}{1 + e^v} = 0.$$

Integrating,

$$\ln x + \ln e^v - \ln(1 + e^v) = C,$$
$$xe^v = A(1 + e^v).$$

Making the reverse substitution,

$$xe^{y/x} = A(1 + e^{y/x}).$$

Exercise 103

GROUP A

Solve each of the following equations.

1. $y^2\, dx + (xy + x^2)\, dy = 0.$
2. $x^2y\, dx - (x^3 + y^3)\, dy = 0.$
3. $(y - 2x)\, dx + (4y + 3x)\, dy = 0.$
4. $(x^2 + y^2)\, dx - 2xy\, dy = 0.$
5. $(xe^{y/x} + y)\, dx - x\, dy = 0.$
6. $(y^2 - xy)\, dx + x^2\, dy = 0.$
7. $(2x^2y + y^3)\, dx - x^3\, dy = 0.$
8. $y^3\, dx + x^3\, dy = 0.$

9. Find the differential equation of the system of ellipses having the points (2,0) and (−2,0) as the extremities of an axis.

10. Find the differential equation of the system of hyperbolas having the coordinates axes as asymptotes.

Solve each of the following equations.

11. $xy\ dx + (x^2 + 1)\ dy = 0$.

12. $(2y^2 + xy^2)\ dx - x^3\ dy = 0$.

13. $xy(dx - dy) = y\ dx + x\ dy$.

14. $\sin x \cos y\ dx - \cos x \sin y\ dy = 0$.

15. $\sin x \cos^2 y\ dx + \cos^2 x\ dy = 0$.

16. $(x - 2y)\ dx + y\ dy = 0$.

154. Exact Differential Equations.

In a differential equation

$$f_1(x,y)\ dx + f_2(x,y)\ dy = 0,$$

it may happen that the left hand member is the differential of some function of x and y. If it is, let u represent that function. Then

$$du = f_1(x,y)\ dx + f_2(x,y)\ dy$$

and

$$f_1(x,y) = \frac{\partial u}{\partial x}, \quad f_2(x,y) = \frac{\partial u}{\partial y},$$

from Section 137. Hence, the solution may be represented by

$$u = F(x,y) = C.$$

Such equations are called *exact differential equations.*

The solutions of exact differential equations frequently can be obtained by inspection. For example, the solution of the equation

$$x\ dy + y\ dx = 0, \quad \text{is} \quad xy = C.$$

Also, the solution of the equation

$$2xy^3\ dx + 3x^2y^2\ dy + 3x^2\ dx = 0,$$

is

$$x^2y^3 + x^3 = C.$$

Sometimes, a differential equation which is not exact as given, may be made exact by the introduction of a term which is called an *integrating*

factor. For example, the equation

$$y \, dx - x \, dy = y^3 \, dy$$

is not exact. However, multiplying both sides of the equation by the integrating factor $1/y^2$, gives an exact differential equation,

$$\frac{y \, dx - x \, dy}{y^2} = y \, dy.$$

Integrating,

$$\frac{x}{y} = \frac{y^2}{2} + C, \quad y^3 - 2x + By = 0.$$

Exercise 104

GROUP A

Solve each of the following equations.

1. $(x + y) \, dx + (x - y) \, dy = 0.$
2. $(2x + y) \, dx + (x + 2y) \, dy = 0.$
3. $ye^x \, dx + (e^x - 3) \, dy = 0.$
4. $x^4 \, dx = x \, dy - y \, dx.$
5. $(y^2 - 4x^2) \, dx + 2xy \, dy = 0.$
6. $(2xy^2 - 3) \, dx + 2x^2y \, dy = 0.$
7. $(x^2 - y^2) \, dx + 2xy \, dy = 0.$
8. $\cos y \, dx - x \sin y \, dy = x \, dx + y \, dy.$

GROUP B

Solve each of the following equations.

9. $x \, dy - y \, dx = x^2 \, dy.$
10. $4x \, dy + y \, dx = xy^2 \, dy.$
11. $(x^2 + 2y^2) \, dx - xy \, dy = 0.$
12. $(x^3 - y^3) \, dx + xy^2 \, dy = 0.$
13. $(x^2 - xy + y^2) \, dx + x^2 \, dy = 0.$
14. $(x^2 + y^2) \, dy + xy \, dx = 0.$

15. Find the differential equation of the system of curves whose slope at any point equals the ordinate of the point. Find the equation of the curve of the system through the point $(0,1)$.

16. Find the differential equation of the system of curves whose slope at any point (x,y) is equal to e^y. Find the equation of the curve of the system which intersects the line $x - 1 = 0$ at the angle of $45°$.

155. Linear Differential Equations of the First Order.

A differential equation is called a *linear differential equation* when the dependent variable and its derivatives appear in the *first degree only.* The

form of the linear equation of the first order is

$$\frac{dy}{dx} + Py = Q, \qquad dy + Py\, dx = Q\, dx,$$

where P and Q are constants or functions of x alone.

If $Q = 0$, the linear equation assumes the form

$$\frac{dy}{dx} + Py = 0, \quad dy + Py\, dx = 0.$$

Separating variables,

$$\frac{dy}{y} = -P\, dx.$$

Integrating,

$$\ln y = -\int P\, dx + C, \quad y = Ae^{-\int P\, dx},$$

or

$$ye^{\int P\, dx} = A.$$

The differentiation of the latter equation gives

$$e^{\int P\, dx}\,(dy + Py\, dx) = 0.$$

This shows that the term

$$e^{\int P\, dx}$$

is an integrating factor of the equation $dy + Py\, dx = 0$, where $Q = 0$. Since Q is a function of x alone, it is also an integrating factor for the linear equation

$$dy + Py\, dx = Q\, dx,$$

as given above, where Q is not zero.

As a result of the above reasoning, the general linear differential equation of the first order may be integrated by the use of the integrating factor

$$e^{\int P\, dx}$$

as follows:

Multiplying by the integrating factor, we have

$$e^{\int P\, dx}\,(dy + Py\, dx) = e^{\int P\, dx} Q\, dx.$$

Integrating both sides of the equation

$$ye^{\int P\, dx} = \int e^{\int P\, dx} Q\, dx + C.$$

This general solution which has been obtained for the linear differential equation cannot be simplified farther until the functions P and Q are known and the indicated integrations carried out. However, this result may be used as a *formula* to give the solution of an equation when it has been expressed in the linear form with the given functions P and Q.

The linear equation

$$x \, dy = (2x + 2y + 1) \, dx,$$

expressed in the type form is

$$\frac{dy}{dx} - \frac{2}{x} y = \frac{2x + 1}{x}, \quad dy - \frac{2}{x} y \, dx = \frac{2x + 1}{x} dx,$$

where

$$P = - \frac{2}{x}, \quad \text{and} \quad Q = \frac{2x + 1}{x}.$$

Then

$$\int P \, dx = - 2 \int \frac{dx}{x} = \ln \frac{1}{x^2}$$

and the integrating factor is

$$e^{\int P \, dx} = e^{\ln 1/x^2} = \frac{1}{x^2}.$$

Using this factor, the equation becomes

$$\frac{1}{x^2} \left(dy - \frac{2}{x} y \, dx \right) = \frac{2x + 1}{x^3} dx.$$

Integrating both sides of the equation,

$$\frac{y}{x^2} = - \frac{1}{2x^2} (4x + 1) + C,$$

or

$$2y = 2Cx^2 - 4x - 1.$$

Exercise 105

GROUP A

Solve each of the following equations.

1. $x \dfrac{dy}{dx} - y = x^2$.

2. $x \dfrac{dy}{dx} - 2y = x + 1$.

3. $\dfrac{dy}{dx} + y = e^{-x}$.

4. $(x + 1)\dfrac{dy}{dx} - 2y = (x + 1)^2$.

5. $\cos^2 x \dfrac{dy}{dx} + y = 1$.

6. $x(x^2 + 1)\dfrac{dy}{dx} + 4x^2y = 2$.

7. $x^2 \dfrac{dy}{dx} + (1 - 2x)\,y = x^2$.

8. $(x + 1)\dfrac{dy}{dx} - 3y = e^x(x + 1)^3$.

GROUP B

Solve each of the following equations.

9. $(x^2 + 1)\dfrac{dy}{dx} + 2xy = 4x^2$.

10. $(2y - x)\,dx + dy = 0$.

11. $(\sin x \cos y - \sin x)\,dx + \sin y\,dy = 0$.

12. $dy + y \cos x\,dx = \sin 2x\,dx$.

13. $3y\,dx - x\,dy = y^3\,dx$.

14. $(x^2 - y^2)\,dx + 2xy\,dy = 0$.

15. $2xy\,dx + (y^2 - 3x^2)\,dy = 0$.

16. Find the particular solution of $3x^2y\,dx + x^3\,dy = x\,dy + y\,dx$ satisfying the condition $x = 2$, $y = 2$.

Reduce each of the following equations to linear form by the substitutions suggested, and solve.

17. $y\dfrac{dy}{dx} + 2xy^2 - x = 0$.   Let $y^2 = v$.

18. $\sin y \dfrac{dy}{dx} + \sin x\,(\cos y - 1) = 0$.   Let $\cos y = v$.

19. $x\dfrac{dy}{dx} + y - x^3y^6 = 0$.   Let $1/y^5 = v$.

20. $4x\dfrac{dy}{dx} + 3y + e^x x^4 y^5 = 0$.   Let $1/y^4 = v$.

21. $3x(1 - x^2)y^2\dfrac{dy}{dx} + (1 - x^2)y^3 = x^3$.   Let $y^3 = v$.

22. $(x + 1)\dfrac{dy}{dx} - (y + 1) = (x + 1)\sqrt{y + 1}$.   Let $\sqrt{y + 1} = v$.

23. Solve the equation $\sec^2 x \tan y\,dx + \sec^2 y \tan x\,dy = 0$.

24. Solve the equation $(x + y)^2\,dy = 4\,dx$, by substituting $x + y = v$.

156. Differential Equations of Type $\dfrac{d^2y}{dx^2} = f\left(x, \dfrac{dy}{dx}\right)$.

Differential equations of the second order and first degree which do not contain the variable y, have the form

$$\frac{d^2y}{dx^2} = f\left(x, \frac{dy}{dx}\right).$$

Only certain simple differential equations of this type are considered in this section.

The solution of the equation

$$\frac{d^2y}{dx^2} = f(x)$$

is

$$\frac{dy}{dx} = \int f(x)\ dx + C_1, \quad y = \int \int f(x)(dx)^2 + C_1 \int dx + C_2.$$

If

$$f(x) = \frac{1}{x}, \quad \frac{dy}{dx} = \ln x + C_1$$

and

$$y = (x \ln x - x) + C_1 x + C_2.$$

The method to be used for a more general equation of this type is illustrated by means of the following examples.

Given the equation

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} = 2x.$$

The absence of the variable y permits us to reduce this equation to one of first order by making the substitution

$$\frac{dy}{dx} = p.$$

Thus, the given equation reduces to

$$x\frac{dp}{dx} + p = 2x, \quad x\ dp + p\ dx = 2x\ dx.$$

The first member of this equation, being exact, enables us to write the solution

$$xp = x^2 + C_1.$$

Upon replacing p by its value, we have

$$x \frac{dy}{dx} = x^2 + C_1, \quad dy = x \, dx + C_1 \frac{dx}{x}.$$

Hence, the solution is

$$y = \tfrac{1}{2}x^2 + C_1 \ln x + C_2.$$

Given the equation

$$x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + a = 0.$$

Letting

$$p = \frac{dy}{dx},$$

$$x^2 \frac{dp}{dx} + xp + a = 0, \quad \frac{dp}{dx} + \frac{1}{x} p = - \frac{a}{x^2}.$$

This being a linear first order equation, we find the integrating factor,

$$e^{\int dx/x} = e^{\ln x} = x.$$

Then

$$x \left(dp + \frac{p}{x} \, dx \right) = - \frac{a}{x} \, dx$$

and

$$xp = - a \ln x + C_1.$$

Replacing p by its value,

$$x \frac{dy}{dx} = - a \ln x + C_1, \quad dy = - \frac{a}{x} \ln x \, dx + \frac{C_1}{x} \, dx.$$

Hence, the solution is

$$y = - \frac{a}{2} \ln^2 x + C_1 \ln x + C_2.$$

Exercise 106

GROUP A

Solve each of the following equations.

1. $x \frac{d^2y}{dx^2} - x = 1.$

2. $x \frac{d^2y}{dx^2} + \frac{dy}{dx} = 0.$

3. $(x + 1) \frac{d^2y}{dx^2} - \frac{dy}{dx} = 0.$

4. $\dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} = 0.$

5. $x\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = x.$

6. $x^2\dfrac{d^2y}{dx^2} + 2x\dfrac{dy}{dx} = 1.$

7. $x\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} = x^3.$

8. $\dfrac{d^2y}{dx^2}\dfrac{dy}{dx} = 1.$

GROUP B

Solve each of the following equations in which $y' = \dfrac{dy}{dx}$ and $y'' = \dfrac{d^2y}{dx^2}$.

9. $y''y' = x.$
10. $yy'' + 2(y')^2 = 0.$
11. $yy'' - 2(y')^2 = 0.$
12. $y'' = \sqrt{1 - (y')^2}.$
13. $xy' = xy^2 + y.$ Let $vy = 1.$
14. $y^2y' = x + y^3.$ Let $y^3 = v.$
15. $x^3y' = 1 + x^2y.$
16. $y' = \sin x + 2y.$

17. Find the equation of the system of curves such that the normal to the tangent at any point of any curve of the system coincides, in direction, with the line joining the point to the origin.

18. Find the equation of the system of curves such that the tangent at any point of any curve of the system and the normal to the tangent at the same point make an isosceles· triangle with the x-axis.

19. A system of curves each member of which cuts each member of a given system of curves at right angles is called an *orthogonal system*. Hence, if the given system is represented by the equation $F(x,y,dy/dx) = 0$ the orthogonal system is represented by the equation $F(x,y, -dx/dy) = 0.$ Find the equation of the system of curves orthogonal to the system of circles $x^2 + y^2 = a^2$

20. Find the equation of the system of curves orthogonal to the system of circles $x^2 + y^2 - 2ax = 0.$

157. Differential Equations of Type $\dfrac{d^2y}{dx^2} = f\!\left(y, \dfrac{dy}{dx}\right).$

Differential equations of the second order and first degree which do not contain the variable x, have the form

$$\frac{d^2y}{dx^2} = f\!\left(y, \frac{dy}{dx}\right).$$

Only certain simple equations of this type are considered in this section.

The solution of the equation

$$\frac{d^2y}{dx^2} = f(y)$$

can be found in the following manner, due to the absence of the variable x. Since

$$2\left(\frac{dy}{dx}\right) dx = 2\, dy,$$

both members of this equation are multiplied by this factor, giving,

$$2\left(\frac{dy}{dx}\right)\frac{d^2y}{dx^2}\, dx = 2f(y)\, dy.$$

The integral of this equation is

$$\left(\frac{dy}{dx}\right)^2 = \int 2f(y)\, dy + C_1.$$

Letting the integral expressed in the second member be denoted by $F(y)$,

$$\frac{dy}{dx} = \pm\sqrt{F(y) + C_1}.$$

The variables being separable in this equation, the solution is

$$\int \frac{dy}{\sqrt{F(y) + C_1}} = \pm\, x + C_2.$$

A more general equation of the type under consideration is represented by

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0,$$

in which the coefficients a, b and c are *constants*.

To determine the solution of such an equation, we first *assume* that it has the form

$$y = e^{mx},$$

in which m is a constant to be determined. Secondly, we *show* that the solution obtained satisfies the given equation.

Given the equation,

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0.$$

If

$$y = e^{mx}, \quad \frac{dy}{dx} = me^{mx} \quad \text{and} \quad \frac{d^2y}{dx^2} = m^2e^{mx}.$$

Substituting these values in the given equation,

$$e^{mx}(m^2 - 3m + 2) = 0.$$

Since the first factor cannot be zero, the second must be and

$$m^2 - 3m + 2 = (m - 1)(m - 2) = 0,$$

hence

$$m = 1 \quad \text{or} \quad m = 2.$$

At this point the student should verify that a constant times a particular solution of a differential equation is a solution of that equation and also, that the sum of two such solutions is a solution.

Using the values of m obtained above, we write the general solution of the given equation,

$$y = C_1e^x + C_2e^{2x},$$

and show that it satisfies that equation as follows:

$$\frac{dy}{dx} = C_1e^x + 2C_2e^{2x}, \quad \frac{d^2y}{dx^2} = C_1e^x + 4C_2e^{2x}.$$

Hence,

$$(C_1e^x + 4C_2e^{2x}) - 3(C_1e^x + 2C_2e^{2x}) + 2(C_1e^x + C_2e^{2x}) = 0.$$

Thus, we have shown that the solution has been obtained, since it satisfies the given differential equation.

If $y = e^{mx}$ is substituted in the equation

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0,$$

we obtain the equation

$$e^{mx}(am^2 + bm + c) = 0.$$

Here again, since the first factor is different from zero, we are called upon to solve the *characteristic* equation

$$am^2 + bm + c = 0.$$

This being a quadratic equation in the unknown m, there are the three following possibilities as to the nature of the roots:

If the roots are real and unequal, let them be denoted by m_1 and m_2. In this case the solution of the given differential equation is written

$$y = C_1 e^{m_1 x} + C_2 e^{m_2 x}.$$

If the roots are equal, let them be denoted by $m_1 = m_2 = m$. In this case the solution of the given equation is written

$$y = e^{mx}(A + Bx).$$

In showing that this equation satisfies the differential equation, we obtain

$$e^{mx}[B(am^2 + bm + c)x + A(am^2 + bm + c) + B(am^2 + bm + c)] = 0,$$

which justifies the solution.

If the roots are imaginary, let them be denoted by $m_1 = \alpha + \beta i$ and $m_2 = \alpha - \beta i$. In this case the solution of the given equation is written

$$y = C_1 e^{(\alpha+\beta i)x} + C_2 e^{(\alpha-\beta i)x}.$$

However, this solution is more often written in the form

$$y = e^{\alpha x}(A \cos \beta x + B \sin \beta x).$$

Here again, either of these forms of the general solution, for imaginary roots, may be shown to satisfy the given differential equation.

Given the differential equation

$$\frac{d^2 y}{dx^2} + 2 \frac{dy}{dx} + 5y = 0.$$

The characteristic equation is

$$m^2 + 2m + 5 = 0,$$

from which

$$m_1 = -1 + 2i \quad \text{and} \quad m_2 = -1 - 2i.$$

Hence, the two forms of the solution are

$$y = C_1 e^{(-1+2i)x} + C_2 e^{(-1-2i)x}$$

and

$$y = e^{-x}(A \cos 2x + B \sin 2x).$$

Exercise 107

GROUP A

Solve each of the following equations in which $y'' = \frac{d^2 y}{dx^2}$ and $y' = \frac{dy}{dx}$.

    1. $y'' - 4y' + 4y = 0.$
    2. $y'' - 3y' + 2y = 0.$

3. $y'' - 4y' + 13y = 0.$
4. $y'' + 6y' + 9y = 0.$
5. $y'' + y' + y = 0.$
6. $y'' - 3y' - 4y = 0.$
7. $y'' = 4y.$
8. $2y'' + 3y' = 0.$

GROUP B

9. If $\alpha + \beta i$ and $\alpha - \beta i$ are the roots of the characteristic equation of

$$ay'' + by' + cy = 0$$

show that the solution $y = C_1 e^{(\alpha+\beta i)x} + C_2 e^{(\alpha-\beta i)x}$ satisfies the differential equation.

10. Show that the solution $y = e^{\alpha x} (A \cos \beta x + B \sin \beta x)$ satisfies the differential equation given in Problem 9 with the same roots of the characteristic equation.

11. Find the general expression for the distance which a body falls from rest in a vacuum, taking the downward sense as positive and by solving the equation $d^2x/dt^2 = g$.

12. Find the general expression for the distance which a body falls from rest, if the resistance is proportional to the velocity, taking the downward sense as positive and by solving the equation $d^2x/dt^2 = g - k \ (dx/dt)$, where k is the proportionality factor.

13. Find the general expression for the distance which a body moves in a straight line, if the resistance is proportional to the square of the velocity, by solving the equation $d^2x/dt^2 = - k(dx/dt)^2$.

14. A body moves in a straight line so that its acceleration is proportional to the velocity. Find the expression for the distance, it being given that $x = 0$ and $v = v_0$ when $t = 0$.

15. A body is projected straight upward with an initial velocity of 1000 ft. per sec. If the resistance of the air is taken as proportional to the velocity and if proportionality factor k is taken to be 0.01, find how far and for how long a time the body will rise.

Solve each of the following equations in which $y''' = d^3y/dx^3$.

16. $y''' - 3y'' + 2y' = 0.$
17. $y''' - 2y'' - y' + 2y = 0.$
18. $y''' + y'' - 2y = 0.$
19. $y''' - 4y'' + 5y' - 2y = 0.$
20. $y''' - 3y'' + 3y' - y = 0$

I. Trigonometric Relations.

Trigonometric Formulas.

$\sin \theta \csc \theta = 1.$ $\qquad\qquad$ $\cos \theta \sec \theta = 1.$ $\qquad\qquad$ $\tan \theta \cot \theta = 1.$

$\tan \theta = \dfrac{\sin \theta}{\cos \theta}.$ $\qquad\qquad$ $\cot \theta = \dfrac{\cos \theta}{\sin \theta}.$

$\sin^2 \theta + \cos^2 \theta = 1.$ $\qquad$ $1 + \tan^2 \theta = \sec^2 \theta.$ $\qquad$ $1 + \cot^2 \theta = \csc^2 \theta.$

$\sin (A \pm B) = \sin A \cos B \pm \sin B \cos A.$

$\cos (A \pm B) = \cos A \cos B \mp \sin A \sin B.$

$\tan (A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}.$

$\sin 2A = 2 \sin A \cos A.$ $\qquad\qquad$ $\cos 2A = \cos^2 A - \sin^2 A.$

$\tan 2A = \dfrac{2 \tan A}{1 - \tan^2 A}.$

$\sin^2 \dfrac{\theta}{2} = \dfrac{1 - \cos \theta}{2}.$ $\qquad\qquad$ $\cos^2 \dfrac{\theta}{2} = \dfrac{1 + \cos \theta}{2}.$

$\tan \dfrac{\theta}{2} = \dfrac{1 - \cos \theta}{\sin \theta}.$

$\sin x + \sin y = 2 \sin \dfrac{x + y}{2} \cos \dfrac{x - y}{2}.$

$\cos x + \cos y = 2 \cos \dfrac{x + y}{2} \cos \dfrac{x - y}{2}.$

$\sin x - \sin y = 2 \sin \dfrac{x - y}{2} \cos \dfrac{x + y}{2}.$

$\cos x - \cos y = -2 \sin \dfrac{x + y}{2} \sin \dfrac{x - y}{2}.$

Reduction to Acute Angles.

$F\left(\dfrac{n}{2} \pi \pm \theta\right) = \pm F(\theta),$ n is zero or any even positive integer.

$F\left(\dfrac{n}{2} \pi \pm \theta\right) = \pm co\text{-}F(\theta),$ n is any odd positive integer.

Trigonometric Laws.

Law of Sines. $\qquad\qquad$ $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}.$

Law of Cosines.	$a^2 = b^2 + c^2 - 2bc \cos A.$
	$b^2 = a^2 + c^2 - 2ac \cos B.$
	$c^2 = a^2 + b^2 - 2ab \cos C.$

Law of Tangents.

$$\tan \frac{A - B}{2} = \frac{a - b}{a + b} \cot \frac{C}{2}.$$

$$\tan \frac{A - C}{2} = \frac{a - c}{a + c} \cot \frac{B}{2}.$$

$$\tan \frac{B - C}{2} = \frac{b - c}{b + c} \cot \frac{A}{2}.$$

Half-Angle Law. $a + b + c = 2s.$ Radius of inscribed circle is r.

$$\tan \frac{A}{2} = \frac{r}{s - a}. \qquad \tan \frac{B}{2} = \frac{r}{s - b}.$$

$$\tan \frac{C}{2} = \frac{r}{s - c}.$$

Radian Measure of Angles.

Radian-Degree. π radians $= 180°.$

Arc-Angle-Radius. $s = r\theta.$

Sector Area. $S = \dfrac{r^2\theta}{2} = \dfrac{rs}{2}.$

Segment Area. $S = \dfrac{r^2(\theta - \sin \theta)}{2}.$

II. Plane Analytic Geometry Relations.

Rectangular coordinates of any point P are (x,y).

If $\dfrac{P_1P_0}{P_0P_2} = \dfrac{r_1}{r_2}$, $x_0 = \dfrac{r_1x_2 + r_2x_1}{r_1 + r_2},$ $y_0 = \dfrac{r_1y_2 + r_2y_1}{r_1 + r_2}.$

If $r_1 = r_2,$ $x_0 = \dfrac{x_1 + x_2}{2},$ $y_0 = \dfrac{y_1 + y_2}{2}.$

Length $P_1P_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$

Slope of $P_1P_2 = m = \dfrac{y_2 - y_1}{x_2 - x_1}.$

Angle between two lines $= \arctan \dfrac{m_2 - m_1}{1 + m_1m_2}.$

General equation of any line L is $Ax + By + C = 0.$

Slope form: $y = mx + b.$

Intercept form: $\dfrac{x}{a} + \dfrac{y}{b} = 1.$

Normal form: $x \cos \alpha + y \sin \alpha - p = 0.$

Distance from L to P_1: $d = \dfrac{Ax_1 + By_1 + C}{\pm\sqrt{A^2 + B^2}}.$

General equation of any circle is $Ax^2 + Ay^2 + Dx + Ey + F = 0$.

Center at (h,k): $(x - h)^2 + (y - k)^2 = a^2$.

Polar equations: $\rho = a$, $\rho = 2a \cos \theta$, $\rho = 2a \sin \theta$.

Parametric equations: $x = a \cos \theta$, $y = a \sin \theta$.

General equation of any conic with axes parallel or coincident with the coordinate axes is $Ax^2 + Cy^2 + Dx + Ey + F = 0$.

Parabola, if A or $C = 0$.   Vertex (h,k): $(y - k)^2 = 4p(x - h)$,
$$(x - h)^2 = 4p(y - k).$$

Ellipse, if $A \neq C$, having the same sign.   Center (h,k): $\dfrac{(x - h)^2}{a^2} + \dfrac{(y - k)^2}{b^2} = 1$.

Hyperbola, if A and C have opposite signs.   Center (h,k): $\dfrac{(x - h)^2}{a^2} + \dfrac{(y - k)^2}{b^2} = \pm 1$.

Asymptotes: $a(y - k) = \pm b(x - h)$.

General equation of any conic is $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$.

Parabola, if $B^2 - 4AC = 0$.

Ellipse, if $B^2 - 4AC < 0$.

Hyperbola, if $B^2 - 4AC > 0$.

Polar equations of conics having their foci at the pole are
$$\rho(1 \pm e \cos \theta) = ep \quad \text{and} \quad \rho(1 \pm e \sin \theta) = ep.$$

Parabola, if $e = 1$.   Ellipse, if $e < 1$.   Hyperbola, if $e > 1$.

Transformations of coordinates.

Translation of axes: $x = X + h$, $y = Y + k$.

Rotation of axes: $x = x' \cos \theta - y' \sin \theta$, $y = x' \sin \theta + y' \cos \theta$.

Rectangular-polar, pole at the origin and polar axis the x-axis: $x = \rho \cos \theta$, $y = \rho \sin \theta$.

TABLE OF INTEGRALS

I. Forms containing a Linear Binomial Factor.

1. $\displaystyle \int \frac{x\,dx}{ax+b} = \frac{x}{a} - \frac{b}{a^2}\ln(ax+b) + C.$

2. $\displaystyle \int \frac{x\,dx}{(ax+b)^2} = \frac{b}{a^2(ax+b)} + \frac{1}{a^2}\ln(ax+b) + C.$

3. $\displaystyle \int x(ax+b)^n\,dx = \frac{x(ax+b)^{n+1}}{a(n+1)} - \frac{(ax+b)^{n+2}}{a^2(n+1)(n+2)} + C,\ n \neq -1,\ n \neq -2.$

4. $\displaystyle \int \frac{dx}{x(ax+b)} = \frac{1}{b}\ln\frac{x}{ax+b} + C.$

5. $\displaystyle \int \frac{dx}{x(ax+b)^2} = \frac{1}{b(ax+b)} + \frac{1}{b^2}\ln\frac{x}{ax+b} + C.$

6. $\displaystyle \int x\sqrt{ax+b}\,dx = \frac{2(3ax-2b)}{15a^2}(ax+b)^{3/2} + C.$

7. $\displaystyle \int \frac{x\,dx}{\sqrt{ax+b}} = \frac{2(ax-2b)}{3a^2}\sqrt{ax+b} + C.$

8. $\displaystyle \int \frac{dx}{x\sqrt{ax+b}} = \frac{1}{\sqrt{b}}\ln\frac{\sqrt{ax+b}-\sqrt{b}}{\sqrt{ax+b}+\sqrt{b}} + C,$ $\qquad\qquad b > 0.$

$\displaystyle \qquad\qquad = \frac{2}{\sqrt{-b}}\arctan\sqrt{\frac{ax+b}{-b}} + C,$ $\qquad\qquad b < 0.$

9. $\displaystyle \int x^n\sqrt{ax+b}\,dx = \frac{2x^n(ax+b)^{3/2}}{a(2n+3)} - \frac{2bn}{a(2n+3)}\int x^{n-1}\sqrt{ax+b}\,dx,\, 2n+3 \neq 0.$

10. $\displaystyle \int \frac{x^n\,dx}{\sqrt{ax+b}} = \frac{2x^n\sqrt{ax+b}}{a(2n+1)} - \frac{2bn}{a(2n+1)}\int \frac{x^{n-1}\,dx}{\sqrt{ax+b}},$ $\qquad\qquad 2n+1 \neq 0.$

11. $\displaystyle \int \frac{dx}{x^n\sqrt{ax+b}} = -\frac{\sqrt{ax+b}}{b(n-1)x^{n-1}} - \frac{a(2n-3)}{2b(n-1)}\int \frac{dx}{x^{n-1}\sqrt{ax+b}},$ $\qquad\qquad n \neq 1.$

II. Forms containing a Quadratic Binomial Factor.

12. $\displaystyle\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \ln \frac{a + x}{a - x} + C.$

13. $\displaystyle\int \frac{dx}{(ax^2 + b)^2} = \frac{x}{2b(ax^2 + b)} + \frac{1}{2b} \int \frac{dx}{ax^2 + b}.$

14. $\displaystyle\int \frac{dx}{x(ax^2 + b)} = \frac{1}{2b} \ln \frac{x^2}{ax^2 + b} + C.$

15. $\displaystyle\int \sqrt{a^2 - x^2}\, dx = \frac{1}{2} x\sqrt{a^2 - x^2} + \frac{1}{2} a^2 \arcsin \frac{x}{a} + C.$

16. $\displaystyle\int \sqrt{x^2 \pm a^2}\, dx = \tfrac{1}{2}x\sqrt{x^2 \pm a^2} \pm \tfrac{1}{2}a^2 \ln (x + \sqrt{x^2 \pm a^2}) + C.$

17. $\displaystyle\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln (x + \sqrt{x^2 \pm a^2}) + C.$

18. $\displaystyle\int \frac{dx}{x\sqrt{a^2 \pm x^2}} = \frac{1}{a} \ln \frac{x}{a + \sqrt{a^2 \pm x^2}} + C.$

19. $\displaystyle\int \frac{dx}{x\sqrt{x^2 - a^2}} = -\frac{1}{a} \arcsin \frac{a}{x} + C.$

20. $\displaystyle\int \frac{\sqrt{a^2 \pm x^2}}{x}\, dx = \sqrt{a^2 \pm x^2} + a \ln \frac{x}{a + \sqrt{a^2 \pm x^2}} + C.$

21. $\displaystyle\int \frac{\sqrt{x^2 - a^2}}{x}\, dx = \sqrt{x^2 - a^2} + a \arcsin \frac{a}{x} + C.$

22. $\displaystyle\int (a^2 - x^2)^{3/2}\, dx = \frac{1}{4}x(a^2 - x^2)^{3/2} + \frac{3}{8}a^2 x\sqrt{a^2 - x^2} + \frac{3}{8}a^4 \arcsin \frac{x}{a} + C.$

23. $\displaystyle\int (x^2 \pm a^2)^{3/2}\, dx = \tfrac{1}{4}x(x^2 \pm a^2)^{3/2} \pm \tfrac{3}{8}a^2 x\sqrt{x^2 \pm a^2} + \tfrac{3}{8}a^4 \ln (x + \sqrt{x^2 \pm a^2}) + C.$

24. $\displaystyle\int \frac{dx}{(a^2 - x^2)^{3/2}} = \frac{x}{a^2\sqrt{a^2 - x^2}} + C.$

25. $\displaystyle\int \frac{dx}{(x^2 \pm a^2)^{3/2}} = \frac{\pm x}{a^2\sqrt{x^2 \pm a^2}} + C.$

26. $\displaystyle\int x^2\sqrt{a^2 - x^2}\, dx = -\frac{1}{4}x(a^2 - x^2)^{3/2} + \frac{1}{8}a^2 x\sqrt{a^2 - x^2} + \frac{1}{8}a^4 \arcsin \frac{x}{a} + C.$

27. $\displaystyle\int x^2\sqrt{x^2 \pm a^2}\, dx = \tfrac{1}{4}x(x^2 \pm a^2)^{3/2} \mp \tfrac{1}{8}a^2 x\sqrt{x^2 \pm a^2} - \tfrac{1}{8}a^4 \ln(x + \sqrt{x^2 \pm a^2}) + C.$

28. $\displaystyle\int \frac{x^2\,dx}{\sqrt{a^2 - x^2}} = -\frac{1}{2}\,x\sqrt{a^2 - x^2} + \frac{1}{2}\,a^2\arcsin\frac{x}{a} + C.$

29. $\displaystyle\int \frac{x^3\,dx}{\sqrt{a^2 - x^2}} = -x^2\sqrt{a^2 - x^2} - \frac{2}{3}(a^2 - x^2)^{3/2} + C.$

30. $\displaystyle\int \frac{x^2\,dx}{\sqrt{x^2 \pm a^2}} = \tfrac{1}{2}x\sqrt{x^2 \pm a^2} \mp \tfrac{1}{2}a^2\ln(x + \sqrt{x^2 \pm a^2}) + C.$

31. $\displaystyle\int \frac{x^n\,dx}{\sqrt{a^2 - x^2}} = -\frac{x^{n-1}\sqrt{a^2 - x^2}}{n} + \frac{a^2(n-1)}{n}\int \frac{x^{n-2}\,dx}{\sqrt{a^2 - x^2}},\qquad n \neq 0.$

32. $\displaystyle\int \frac{dx}{x^n\sqrt{a^2 - x^2}} = -\frac{\sqrt{a^2 - x^2}}{a^2(n-1)x^{n-1}} + \frac{n-2}{a^2(n-1)}\int \frac{dx}{x^{n-2}\sqrt{a^2 - x^2}},\qquad n \neq 1.$

33. $\displaystyle\int x^n\sqrt{x^2 \pm a^2}\,dx = \frac{x^{n-1}(x^2 \pm a^2)^{3/2}}{n+2} \mp \frac{a^2(n-1)}{n+2}\int x^{n-2}\sqrt{x^2 \pm a^2}\,dx,\ n + 2 \neq 0.$

34. $\displaystyle\int \frac{x^n\,dx}{\sqrt{x^2 \pm a^2}} = \frac{x^{n-1}\sqrt{x^2 \pm a^2}}{n} \mp \frac{a^2(n-1)}{n}\int \frac{x^{n-2}\,dx}{\sqrt{x^2 \pm a^2}},\qquad n \neq 0.$

35. $\displaystyle\int \frac{dx}{x^n\sqrt{x^2 \pm a^2}} = \mp \frac{\sqrt{x^2 \pm a^2}}{a^2(n-1)x^{n-1}} \mp \frac{n-2}{a^2(n-1)}\int \frac{dx}{x^{n-2}\sqrt{x^2 \pm a^2}},\qquad n \neq 1.$

36. $\displaystyle\int \frac{dx}{\sqrt{2ax - x^2}} = 2\arcsin\sqrt{\frac{x}{2a}} + C.$

37. $\displaystyle\int \frac{x^n\,dx}{\sqrt{2ax - x^2}} = \frac{x^{n-1}\sqrt{2ax - x^2}}{n} + \frac{a(2n-1)}{n}\int \frac{x^{n-1}\,dx}{\sqrt{2ax - x^2}},\qquad n \neq 0.$

38. $\displaystyle\int \sqrt{2ax - x^2}\,dx = \frac{1}{2}(x - a)\sqrt{2ax - x^2} + \frac{1}{2}a^2\arcsin\frac{x - a}{a} + C.$

III. Trigonometric Functions.

39. $\displaystyle\int \sin^2 ax\,dx = \frac{x}{2} - \frac{\sin 2ax}{4a} + C.$

40. $\displaystyle\int \cos^2 ax\,dx = \frac{x}{2} + \frac{\sin 2ax}{4a} + C.$

41. $\displaystyle\int \sin^n ax\,dx = -\frac{1}{an}\sin^{n-1}ax\cos ax + \frac{n-1}{n}\int \sin^{n-2}ax\,dx.$

42. $\displaystyle\int \cos^n ax\, dx = \frac{1}{an} \cos^{n-1} ax \sin ax + \frac{n-1}{n} \int \cos^{n-2} ax\, dx.$

43. $\displaystyle\int \sin^m ax \cos^n ax\, dx =$

$$\frac{\sin^{m+1} ax \cos^{n-1} ax}{a(m+n)} + \frac{n-1}{m+n} \int \sin^m ax \cos^{n-2} ax\, dx, \; m+n \neq 0.$$

44. $\displaystyle\int \sin^m ax \cos^n ax\, dx =$

$$-\frac{\sin^{m-1} ax \cos^{n+1} ax}{a(m+n)} + \frac{m-}{m+n} \int \sin^{m-2} ax \cos^n ax\, dx, \; m+n \neq 0.$$

45. $\displaystyle\int \sin^m ax \cos^n ax\, dx =$

$$-\frac{\sin^{m+1} ax \cos^{n+1} ax}{a(n+1)} + \frac{m+n+2}{n+1} \int \sin^m ax \cos^{n+2} ax\, dx, \; n \neq -1.$$

46. $\displaystyle\int \tan ax\, dx = \frac{1}{a} \ln \sec ax + C.$

47. $\displaystyle\int \cot ax\, dx = \frac{1}{a} \ln \sin ax + C.$

48. $\displaystyle\int \tan^2 ax\, dx = \frac{1}{a} \tan ax - x + C.$

49. $\displaystyle\int \cot^2 ax\, dx = -\frac{1}{a} \cot ax - x + C.$

50. $\displaystyle\int \tan^n ax\, dx = \frac{1}{a(n-1)} \tan^{n-1} ax - \int \tan^{n-2} ax\, dx, \qquad\qquad n \neq 1.$

51. $\displaystyle\int \cot^n ax\, dx = -\frac{1}{a(n-1)} \cot^{n-1} ax - \int \cot^{n-2} ax\, dx, \qquad\qquad n \neq 1.$

52. $\displaystyle\int \sec ax\, dx = \frac{1}{a} \ln(\sec ax + \tan ax) + C = \frac{1}{2a} \ln \frac{1+\sin ax}{1-\sin ax} + C.$

53. $\displaystyle\int \csc ax\, dx = \frac{1}{a} \ln (\csc ax - \cot ax) + C = \frac{1}{a} \ln \tan \frac{ax}{2} + C.$

54. $\displaystyle\int \sec^3 ax\, dx = \frac{1}{2a} \frac{\sin ax}{\cos^2 ax} + \frac{1}{4a} \ln \frac{1+\sin ax}{1-\sin ax} + C.$

55. $\displaystyle\int \csc^3 ax\, dx = -\frac{1}{2a} \frac{\cos ax}{\sin^2 ax} + \frac{1}{4a} \ln \frac{1-\cos ax}{1+\cos ax} + C.$

56. $\displaystyle\int \sec^n ax\, dx = \frac{\tan ax \sec^{n-2} ax}{a(n-1)} + \frac{n-2}{n-1} \int \sec^{n-2} ax\, dx,$ $n \neq 1.$

57. $\displaystyle\int \csc^n ax\, dx = - \frac{\cot ax \csc^{n-2} ax}{a(n-1)} + \frac{n-2}{n-1} \int \csc^{n-2} ax\, dx,$ $n \neq 1.$

58. $\displaystyle\int x \sin ax\, dx = \frac{1}{a^2} \sin ax - \frac{1}{a} x \cos ax + C.$

59. $\displaystyle\int x \cos ax\, dx = \frac{1}{a^2} \cos ax + \frac{1}{a} x \sin ax + C.$

60. $\displaystyle\int x^n \sin ax\, dx = - \frac{1}{a} x^n \cos ax + \frac{n}{a} \int x^{n-1} \cos ax\, dx.$

61. $\displaystyle\int x^n \cos ax\, dx = \frac{1}{a} x^n \sin ax - \frac{n}{a} \int x^{n-1} \sin ax\, dx.$

62. $\displaystyle\int x \sin^2 ax\, dx = \frac{1}{4} x^2 - \frac{1}{4a} x \sin 2ax - \frac{1}{8a^2} \cos 2ax + C.$

63. $\displaystyle\int x \cos^2 ax\, dx = \frac{1}{4} x^2 + \frac{1}{4a} x \sin 2ax + \frac{1}{8a^2} \cos 2ax + C.$

64. $\displaystyle\int \sin ax \sin bx\, dx = \frac{\sin (a-b)x}{2(a-b)} - \frac{\sin (a+b)x}{2(a+b)} + C,$ $a \neq b.$

65. $\displaystyle\int \sin ax \cos bx\, dx = - \frac{\cos (a-b)x}{2(a-b)} - \frac{\cos (a+b)x}{2(a+b)} + C,$ $a \neq b.$

66. $\displaystyle\int \cos ax \cos bx\, dx = \frac{\sin (a-b)x}{2(a-b)} + \frac{\sin (a+b)x}{2(a+b)} + C,$ $a \neq b.$

IV. Exponential Forms.

67. $\displaystyle\int x e^{ax}\, dx = \frac{1}{a^2} (ax - 1) e^{ax} + C.$

68. $\displaystyle\int x^2 e^{ax}\, dx = \frac{1}{a^3} (a^2 x^2 - 2ax + 2) e^{ax} + C.$

69. $\displaystyle\int x^n e^{ax}\, dx = \frac{1}{a} x^n e^{ax} - \frac{n}{a} \int x^{n-1} e^{ax}\, dx.$

70. $\displaystyle\int e^{ax} \sin bx\, dx = \frac{e^{ax}(a \sin bx - b \cos bx)}{a^2 + b^2} + C.$

71. $\displaystyle\int e^{ax} \cos bx \, dx = \frac{e^{ax}(b \sin bx + a \cos bx)}{a^2 + b^2} + C.$

72. $\displaystyle\int \ln x \, dx = x \ln x - x + C.$

73. $\displaystyle\int x^n \ln x \, dx = x^{n+1}\left[\frac{\ln x}{n + 1} - \frac{1}{(n + 1)^2}\right] + C,$ $n \neq -1.$

74. $\displaystyle\int \ln^n x \, dx = x \ln^n x - n \int \ln^{n-1} x \, dx.$

V. Inverse Trigonometric Functions.

75. $\displaystyle\int \arcsin \frac{x}{a} \, dx = x \arcsin \frac{x}{a} + \sqrt{a^2 - x^2} + C.$

76. $\displaystyle\int \arcsin^2 \frac{x}{a} \, dx = x \arcsin^2 \frac{x}{a} - 2ax + 2\sqrt{a^2 - x^2} \arcsin \frac{x}{a} + C.$

77. $\displaystyle\int x \arcsin \frac{x}{a} \, dx = \frac{1}{4}(2x^2 - a^2) \arcsin \frac{x}{a} + \frac{x}{4}\sqrt{a^2 - x^2} + C.$

78. $\displaystyle\int x^n \arcsin \frac{x}{a} \, dx = \frac{x^{n+1}}{n + 1} \arcsin \frac{x}{a} - \frac{1}{n + 1}\int \frac{x^{n+1} \, dx}{\sqrt{a^2 - x^2}},$ $n \neq -1.$

79. $\displaystyle\int \arccos \frac{x}{a} \, dx = x \arccos \frac{x}{a} - \sqrt{a^2 - x^2} + C.$

80. $\displaystyle\int x^n \arccos \frac{x}{a} \, dx = \frac{x^{n+1}}{n + 1} \arccos \frac{x}{a} + \frac{1}{n + 1}\int \frac{x^{n+1} \, dx}{\sqrt{a^2 - x^2}},$ $n \neq -1.$

81. $\displaystyle\int \arctan \frac{x}{a} \, dx = x \arctan \frac{x}{a} - \frac{a}{2} \ln(a^2 + x^2) + C.$

82. $\displaystyle\int x \arctan \frac{x}{a} \, dx = \frac{1}{2}(a^2 + x^2) \arctan \frac{x}{a} - \frac{ax}{2} + C.$

83. $\displaystyle\int x^n \arctan \frac{x}{a} \, dx = \frac{x^{n+1}}{n + 1} \arctan \frac{x}{a} - \frac{a}{n + 1}\int \frac{x^{n+1} \, dx}{a^2 + x^2},$ $n \neq -1.$

INDEX

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ANSWERS

Pages 4, 5, Ex. 1

1. $6x^2$.

3. $S = \dfrac{y^2}{4\pi} \cdot$

5. $2x = 3y$.

7. $-3, 21, -19, -x^3 - x^2 - 2x - 3$.

9. $y = \dfrac{1 \pm \sqrt{5}}{2} \, x$.

11. $2, 6$.

13. $\dfrac{\pi}{6} y^3$.

15. $\dfrac{1}{6} \sqrt{\dfrac{S^3}{\pi}} \cdot$

17. $v = ks^2$.

19. $1, 0, \cos\theta, \sin\theta$.

27. $S = 32 \sin\theta \cos\theta = 16 \sin 2\theta$.

29. $S = \dfrac{a}{b}(bx - x^2)$ or $\dfrac{b}{a}(ax - x^2)$.

31. $V = 2\pi x^2 \sqrt{r^2 - x^2}$.

33. $-1, 0, 1, 1$.

Pages 9, 10, Ex. 2

1. 1.

3. none.

5. 0.

7. 0.

11. 0.

13. ± 1.

15. $\frac{1}{2}$.

17. none, none.

21. 2.

23. $2x$.

25. $1 - 3x^2$.

Pages 13, 14, Ex. 3

7. $x = 2$.

9. $-1, -2, 3$.

15. 2.

17. $1, 2, 3$.

19. -4 and -3, -2 and -1, 1 and 2.

23. $x = \cdots, -\dfrac{3\pi}{4}, -\dfrac{\pi}{4}, \dfrac{\pi}{4}, \dfrac{3\pi}{4} \cdots$.

Pages 19, 20, Ex. 4

1. $4t\,\Delta t + 2\overline{\Delta t}^2 - \Delta t, \ 4t + 2\Delta t - 1$.

5. $1 - 6t - 3\Delta t, \ -5.9$.

5. 14 units/sec.

7. $4x\,\Delta x + 2\overline{\Delta x}^2 - \Delta x, \ 4x + 2\Delta x - 1,$
$4x - 1$.

9. $2, 0, -2$.

11. $(3x^2 - 2x + 1)\,\Delta x + (3x - 1)\,\overline{\Delta x}^2 +$
$\overline{\Delta x}^3, \ (3x^2 - 2x + 1) + (3x - 1)\,\Delta x$
$+ \overline{\Delta x}^2, \ (3x^2 - 2x + 1)$.

13. $3 - 3x^2, \ -72$.

15. 2π.

19. $a^2/2$.

Pages 21, 22, Ex. 5

1. $6 + 3x^2$.

3. $3x^2 - 4x + 1$.

5. $-1/x^2$.

7. $(3, -9)$.

9. 5 sq. ins./in.

11. $1 + \dfrac{2}{x^2} \cdot$

13. $2x - \dfrac{3}{x^2} \cdot$

15. $4x^3 - 1$.

17. $\sqrt{3} \, x/2$.

19. $-6, 2$.

21. $1/(x + 1)^2$.

23. $1/2\sqrt{x}$.

25. $1/2\sqrt{x - 1}$.

27. $\pi(2r + 5)$.

29. -0.5 cu. in./lb./sq.in.

Pages 24, 25, Ex. 6

1. $6(x^2 - x + 1)$.

3. $-(3x^2 + 6x + 2)$.

5. $-1/x^2$.

7. $3\sqrt{x} - \dfrac{3}{2\sqrt{x}} \cdot$

9. $-\left(\dfrac{8}{x^3} + \dfrac{3}{\sqrt{x}} + \dfrac{1}{\sqrt{x^3}}\right) \cdot$

11. $4x(x^2 - 1)$.

13. $5\sqrt{x^3}/2 - 1$.

15. $9\sqrt{x}/2 - 5\sqrt{x^3}$.

17. 2, 3.

19. 1, 2, 3.

Pages 27, 28, Ex. 7

1. $2x - y - 6 = 0$.

3. $11x - y - 16 = 0$.

5. $4x - 3y - 6 = 0$.

7. $-6, 6$.

9. $x - 5y - 16 = 0$.

11. $(-1, -16)$, $(1/2, -121/16)$, $(1, -8)$,
$y + 16 = 0$, $16y + 121 = 0$,
$y + 8 = 0$.

13. 45°, arctan 3.

15. arctan $(2/5)$.

17. $15x - y - 38 = 0$,
$15x - y + 70 = 0$.

19. $3x + y - 16 = 0$.

Pages 32, 33, Ex. 8

1. 14, -94.

3. $(2, -22)$, $(-4, 86)$, $x < -4$
and $2 < x$.

5. $(0, 1)$, $(1, 2)$.

11. $p = 20x - x^2$, $x < 10$, $10 < x$,
10×10.

13. $4/3 < x$, $x < 4/3$.

15. $2 < x$, $x < 2$.

17. 5625 ft., 150 ft., arctan (-150).

19. arctan $(2/9)$.

21. $(-2, -4)$, $2\sqrt{14}$.

23. $\left(-\dfrac{b}{2a}, \ -\dfrac{b^2 - 4ac}{4a}\right) \cdot$

Pages 35, 36, Ex. 9

1. 11.

3. -10, -37 minimum.

5. -3.

11. $f(-3) = -4$, $f(2) = -129$.

15. 6, -6.

17. $-4, 8$.

19. $1/2$.

Pages 40–42, Ex. 10

1. $(1, -7)$ min., $(-3/2, 97/4)$ max.

3. $3 < x$ upward, $x < 3$ downward.

9. $3/2 < t$, $t < 3/2$.

17. $f'(a) = +$, $f''(a) = -$; $f'(b) = -$,
$f''(b) = +$; $f'(c) = +$,
$f''(c) = +$; $f'(d) = -$,
$f''(d) = -$.

19. $f'(a) = -$, $f''(a) = 0$, $f'(b) = +$,
$f''(b) = 0$; $f'(c) = f''(c) = 0$.

21. $(0, 0)$, $(\pm 1, \pm 2/5)$,
$(\pm \sqrt{2}/2, \pm 7\sqrt{2}/40)$.

23. $4y = x^3 - 12x + 16$.

25. $2y = x^3 - 5x^2 + 2x + 8$.

27. $f(x) = 2 - 6x^2 + 6x^3 - 2x^4$.

Pages 44–46, Ex. 11

1. 16 cu. ins.

3. 36×48 ins.

5. $2x + y - 6 = 0$.

9. Side base = twice height.

11. Height = radius base.

13. $1 \times 1 \times 2$ ft.; circum. = 4 ft.,
length = 2 ft.

15. $\pi ab^2/27$.

17. $\frac{1}{2}$ in., 4 ins.

19. 5 ins.

21. a/π ins.

23. 10/9, 4 ft. 9 ins.

25. $2\frac{8}{17}$ hrs.

Pages 49, 50, Ex. 12

1. $4, 2 + 4a, a$; $0.4, 0.02 + 0.4a, 0.01a$;
$0.04, 0.0002 + 0.04a, 0\,0001a$.

3. 1st, 2nd and 1st.

5. 1st.

7. 2nd.

9. 1st, $-6x$.

11. 2nd, ax^2.

13. 1st, $3ax$.

15. 1st, $-2(x + 1)\,\Delta x$.

17. 1st, $2(2x^3 + 1)\,\Delta x$.

19. 1st.

21. 2nd.

23. 1st.

25. 3rd.

Page 53, Ex. 13

1. $2\Delta x - 2x\,\Delta x - \overline{\Delta x}^2$,
$2(1 - x)\,dx, -\overline{\Delta x}^2$.

3. $(3x^2 - 2x)\,dx, (3x\,\Delta x + \overline{\Delta x}^2 - \Delta x)\,\Delta x$.

5. $\pi(2r + \Delta r)\,\Delta r, 2\pi r\,dr$.

7. $-0.0199, -0.02$.

9. $(8x^3 - 12x)\,dx$.

13. $\dfrac{\sqrt{x} - 6x^3}{2x}\,dx$.

15. $\dfrac{2x - 4 - 3\sqrt{x}}{2x^3}\,dx$.

17. $(2t - 9t^2)\,dt$.

21. $0.0004\pi = 0.001257$ sq. in.

Page 55, Ex. 14

1. $t/2$.

3. $3/2t$.

5. $r^2(1 - 6r)$.

7. $3x - 2y = 0$.

9. $7x + 2y - 11 = 0, 5x + 2y - 5 = 0$.

Pages 57–59, Ex. 15

1. 1.257893, 1.256636 sq. ins.

3. 27.054 cu. ins.

5. 0.0067, 0.67%.

7. 14.034.

9. 10.04.

11. 16.024, 24.044, 44.048.

13. $2003\pi/3$ cu. ins., $100.1(\sqrt{5} + 1)\pi$ sq. ins.

15. $-dx/x^2$.

17. Decreased 23 5 cu. ins.

19. 33.004.

21. 0.03.

23. $0 < t < 3, 8 < t < 12; 1 < t; t < 1$.

25. $21.60.

Pages 62, 63, Ex. 16

1. $x^3 - 3x^2 + 2x + C$.

3. $ax + bx^2/2 + C$.

5. $5(x + 3)^3/3 + C$.

7. $2x^3/3 - x^2/2 + x + C$.

9. $9x^4/4 + 5x^3/3 - 4x^2 + 3x + C$.

11. $3t^4/4 - 2t^3/3 + 3t^2 + 8t + C$.

13. $x^2/2 - x + 2x^{3/2}/3 + C$.

15. $5x^4/2 - x^5 + C$.

17. $y = x^2 - x + C$.

19. $s = 2t^3/3 - 3t^2/2 + 6t + 2$.

Pages 66, 67, Ex. 17

1. $y = x^3 - x^2 - x + C.$
3. $s = 16t^2.$
5. 21 units.
7. 1/6 unit.
9. $3y = 4x^{3/2} - 14.$

11. $y = 27x - x^3 - 26.$
13. $(y - C)^2 = 4x,$ all tan to x-axis.
15. 2 secs., 16 ft.
17. $t = 4, v = -12,$ 32 units.

Pages 71, 72, Ex. 18

1. 15.
3. 32/3.
5. 27/4.
7. 15/2.
9. 1/2.
11. 4/3.
13. 72.

15. 128/3.
17. $y = x^3 - x^2 - x - 2.$
19. $1 < t < 4,$ 9/2.
21. 40 cents.
23. 0.5%.
25. 32/3.

Pages 78–80, Ex. 19

1. $8x^3 - 3x^2 + 2.$
3. $-2/(x - 2)^2.$
5. $18x(3x^2 - 2)^2.$
7. $2(4x^3 - 3x^2 - x + 2).$
9. $-4x/(1 + x^2)^2.$
11. $(1/3, 8/27), (1, 0), (2/3, 2/27).$
13. $2\sqrt{3}/3.$
17. $(2 + 3x)/(2 - 3x)^3.$
19. $2(2x - 1)(2x + 1)^2(10x - 1).$
21. $\sqrt{1 - x}\,(4x - 7x^2 - 6)/2.$
23. $-3/\sqrt{(3 - x)(3 + x)^3}.$

25. $-(2t^2 + 1)/t^2.$
27. $x + 2y = 0, 2x + y - 9 = 0.$
29. $1 \times 4,$ maximum.
31. $2(2t^5 + 2t^3 - 3t^2 - 1).$
33. $(b^2 + 2z^2)/(b^2 - z^2)^{5/2}.$
35. $(y^2 + 3by - 2b^2)/2(y + b)^2\sqrt{y - b}.$
37. $2b(a - 2x)/x^3(x - a)^3.$
39. $-2ax^2/(x^3 + a)^{2/3}(x^3 - a)^{4/3}.$
41. $9 - 2\sqrt{3}$ mi.
43. $-(1 + t^2)^2/4t^{3/2}.$
45. $t = 2, 1.$

Pages 82, 83, Ex. 20

1. $y = (7 - x)/3.$
3. $y = 4x/(x + 2).$
5. $-y/x.$
7. $(2x - y + 2)/(x - 2y).$
9. $(2x + y)/(2y - x).$
11. $3x + 4y - 25 = 0,$
   $3x - 4y + 25 = 0.$
13. $x + 6y - 13 = 0.$
15. $\dfrac{1 - y^2}{xy} = \dfrac{\pm 2}{x^2\sqrt{x^2 + 2}}.$

17. $-b^2x/a^2y.$
19. $-\sqrt{y}/\sqrt{x}.$
21. $x + 3y - 13 = 0.$
23. $3x - y - 2 = 0, 3x + y - 2 = 0.$
25. $xx_1 + yy_1 = a^2.$
27. $\pi/2,$ arctan 7.
29. $-(y + a)/(x + b).$
31. $(y^2 + a)/(3y^2 - 2xy - b).$
33. $\dfrac{x + \sqrt{x^2 + y^2}}{2y\sqrt{x^2 + y^2} - y}.$

Page 85, Ex. 21

1. $2a/y$, $-4a^2/y^3$.
3. $-y/x$, $2y/x^2$.
5. $-x/y$, $-a^2/y^3$.
7. $-(y+3)/(x+2)$,
   $-2(y+3)/(x+2)^2$.
9. $-y/x$, $-2y/x^2$.
11. $3t/2$, $3/4t$.
13. $-\sqrt{y/x}$, $\sqrt{a}/2x^{3/2}$.

15. $-3a^2x/y^5$.
17. y/x.
19. $-120/(2y-1)^7$.
21. $(1, 1)$ min., $(-1, -1)$ max.
23. $(-1, -2)$, $(1, 2)$.
25. $(1, 1)$ max.
27. $128, 8\sqrt[4]{2}$.

Pages 87, 88, Ex. 22

1. Base = side.
3. $\sqrt{2}$.
5. $a/2 \times a\sqrt{3}/2$, a = diameter.
7. $r = \sqrt{2}h$.
9. $e = 2h$
11. $a\sqrt{2} \times b\sqrt{2}$.

13. $3\sqrt{3}ab/4$.
15. $(0, 2)$ max., $(\pm 2\sqrt{3}/3, 3/2)$.
17. Length = 2 diameter.
19. $2\sqrt{3}$ mi.
21. $3a/4$.

Page 94, Ex. 23

9. $x - 4y = 0$, $y^2 - x + 2y + 1 = 0$.
11. $x + 2y - 3 = 0$,
    $3y^2 - 4x - 14y + 15 = 0$.

23. $t/2$, $1/12t$.
25. $-(1+t)^2/(1-t)^2$,
    $4(1+t)^3/(1-t)^3$.

Pages 99, 100, Ex. 24

1. $2, -4, 2\sqrt{5}, (2, -1)$.
3. $2, -2, 2\sqrt{2}, (4, 0)$.
5. $6\sqrt{t^2 - 2t + 2}$, $t = 1$, $(3, 2)$.
7. $\arctan(2/3t)$, $\arctan(1/3t)$, $y^3 = x^2$.
9. $\arctan(4/3)$, $1/2$ hr.
11. $2, 12, 2\sqrt{37}$, $\arctan 6$; $0, 12, \pi/2$.

13. $2, -2, 2\sqrt{2}, 3\pi/4$; $0, 6, \pi/2$.
15. $1/4, -1, \sqrt{17}/4$, $\arctan -4$;
    $-1/4, 2, \sqrt{65}/4$, $\arctan -8$.
17. $-3/2, 3/2, 3\sqrt{2}/2, 3\pi/4$; $-9/2$,
    $-15/2, 3\sqrt{34}/2$, $\arctan(5/3)$.
19. $\pm 2\sqrt{5}, \pm\sqrt{5}, \pm 1/3, \mp 2/3$.

Pages 101–103, Ex. 25

1. $x = 100\sqrt{2}\,t$, $y = 100\sqrt{2}\,t - 16t^2$;
   $141.4, 45.4, 148.8$.
3. $5\sqrt{6}/2$ secs., $200\sqrt{6}$ ft.

7. $2v_0^2y = 2v_0^2x \tan\alpha - gx^2 \sec^2\alpha$.
11. $\pi/4$.
19. $12/5, -16/5$.

Pages 105, 106, Ex. 26

1. $1/25\pi$ ft./min.
3. $2/\pi$ ft./min.
5. 50 mi./hr.
7. $2\sqrt{5}$ ft./sec.
9. $80/7, 24/7$ ft./sec.
11. $2\sqrt{3}/15$ ft./hr.

13. $\dfrac{1500\pi^2}{\sqrt{25\pi^2 + 1}}$ ft./min.
15. $1\,2$ sq. ft./min.
17. $-250/\sqrt{89}, 250/\sqrt{41}$ ins./sec., $t = 1$.
19. $\dfrac{35(70t \pm \sqrt{3})}{\sqrt{1225t^2 \pm 35\sqrt{3}\,t + 1}}$ mi./hr.

Pages 108, 109, Ex. 27

1. 2000 cu. ft.
3. 80, 120 mi.
5. $x = 2t^2 + 3t$, $y = 6t$.
7. $(y - 2)^2 = x^3$, $2\sqrt{10}$.
9. arctan $(44/15)$, 50.3 mi./hr.
11. $6s = t^3 - 3t^2 - 6t$, 2.7 secs., 4.4 secs.
13. $-2kxv_x/a$.

15. $x = kt^2/2 + at$, $y = bt$.
17. $(2t + 9)^{3/2}/3$, $y = 2t + 9$.
19. 422 rds., 6 mins.
21. $50 + 10\sqrt{3}$ rds.
23. $(\pm 8\sqrt{5}/5, \pm 2\sqrt{5}/5)$.
25. 3 hrs., 60 mi.
27. 8.155.

Pages 111, 112, Ex. 28

1. $y = x^2 + C$, 8.
3. 8.
5. $128/3$.
7. $33/2$.
9. $44/3$.

11. $12t + 8t^2 + C$, 204.
13. $14/3$.
15. $1/12$.
17. $6x$, 15.

Pages 113, 114, Ex. 29

1. $21/2$.
3. $486/5$.
5. 7.

7. $3/4$.
9. $16/3$.
11. 36.

13. 1.
15. 72.
17. $64/3$.
19. $x + y = 2a$.

Page 115, Ex. 30

1. 90.
3. 36.

5. $25/12$.
7. 5.85312.

9. 3.58433.

Page 117, Ex. 31

1. $0.86157 < 1 < 1.16157$.
3. $0.66877 < 0.69315 < 0.71877$.

5. $0.7600 < 0.7854 < 0.8100$.
7. $1.8390 < 2 < 2.153 \overset{,}{}$

Page 123, Ex. 32

1. $128/3$.
3. $9/2$.

5. $71/6$.
7. $32\sqrt{2}/15$.

9. $1/3$.
11. $40/3$.

13. $6\sqrt{3}$.
15. $64/3$.

Pages 128, 129, Ex. 33

1. 72π.
3. π.
5. $128\pi/105$.
7. $384\pi/5$.

9. $1392\pi/5$.
11. 16π.
13. $128\pi/35$.
15. $14/3$.

17. $\pi r^2 h/3$.
19. $4\pi r^3/3$.
21. $\pi h(3r_1^2 + 3r_2^2 + h^2)/6$.
23. 16, $384\pi/105$.

Pages 131, 132, Ex. 34

1. $288\sqrt{3}$.
3. 576.
5. 288.
7. 144.

9. $648\sqrt{3}$.
11. 972π.
13. 800.
15. $8a^3/3$.

17. $64/5$.
19. $a^2 h/3$.
21. $a \times a\sqrt{3}$.

Page 136, Ex. 35

1. $640w/3$.
3. $320w/3$.
5. 4 ft. below base.
7. $8{,}586w$.

9. $50{,}500w/3$.
11. $16{,}384w/5$.
13. $21{,}504w/5$.

15. $53{,}248w/15$.
17. $8\sqrt[5]{8}$ below vertex.
19. 0.8 cu. in./min.

Pages 138–140, Ex. 36

1. 6.
3. $4000aB/(4000 + a)$.
5. $3/16$, 3 ft. lbs.
7. $5400\pi w$ ft. lbs.
9. $135\pi w$ ft. lbs.

11. $144\pi w$ ft. lbs.
13. $393\pi w$ ft. lbs.
15. 0.3333485.
17. Not at $x = -1$, no, $x > -1$.

19. $18t + 6$, 0, 18.
21. $3\sqrt{5}$.
23. $660w$, 11 ft.
25. 5050 ft. lbs.

Pages 143, 144, Ex. 37

1. $2\pi/3$.
3. $\pi/2$.
5. 2π.

7. 4π.
9. 2π.

11. 400π ins./sec., 100π ins./sec.
23. 6.2838 sq. ins., 0.1278 sq. in.

Pages 149, 150, Ex. 38

1. $18 \cos 3x$.
3. $6 \sin (1 - 2x)$.
5. $\sin x \cos^2 x (2 \cos^2 x - 3 \sin^2 x)$.
7. $-12 \cot^2 4x \csc^2 4x$.
9. $1/2$, $-1/2$, -1.
11. 2, 2, 8.
13. 1, 0, -1, $-\sqrt{2}$, -1.
15. $-4(1 - \sin 2x) \cos 2x$.
17. $\sin x/(2\sqrt{1 - \cos x})$.

19. $(x \cos x - \sin x)/x^2$.
21. $2 \sec x(\tan x - \sec x)^2$.
23. $\cos x$, $-f(x)$, $-f'(x)$, $f(x)$.
25. $\cos x/\sin y$.
27. $\dfrac{\sec^2 (x + y) + \sec^2 (x - y)}{\sec^2 (x - y) - \sec^2 (x + y)}$.
29. y/x.
31. $-\cot \theta$, $-(1/4) \csc^3 \theta$.
33. $(b/a) \csc \theta$, $-(b/a^2) \cot^3 \theta$.
35. $\pi/4$.

Pages 151, 152, Ex. 39

1. $2x - y + 1 = 0$,
$6x - 12y = \pi - 12\sqrt{3}$.
3. $0 < x < \pi/2$, $\pi/2 < x < \pi$.
5. $\pi/2 < x < \pi$, $0 < x < \pi/2$.
9. $\sqrt{5}$, $-\sqrt{5}$.
11. $3\sqrt{3}/2$, $-3\sqrt{3}/2$.
13. $0 < x < \pi/4$, $5\pi/4 < x < 2\pi$;
$\pi/4 < x < 5\pi/4$.

15. 4 sq. ins./min.
19. $\arctan (6/17)$.
21. $13\sqrt{13}$ ft.
23. $(-2, 3)$, 2; $(2, 5)$, 4.
25. 0.86748, 0.00167.

Pages 155, 156, Ex. 40

1. Amplitude 4, period π.
3. $s = -5 \sin(\pi t/4)$, $-5\pi/4$.
5. 0, $9\sqrt{2}$, 18, 18; 27, $27\sqrt{2}/2$, 0, 0.
7. $s = 4 \sin (t/4)$.

9. $j/s = -16$, $\pi/2$, 2.
11. $j/s = -1/9$, 5/3, 0.
13. 5 ft.
15. $8\pi a^2 b/15$.

Page 158, Ex. 41

1. $x = \dfrac{1}{3}\cos\dfrac{y}{2}$.

3. $x = \sin y$.

15. $x = \dfrac{1}{2}\sin y,\ \dfrac{\pm 2}{\sqrt{1 - 4x^2}}$.

17. $x = 2\tan y,\ \dfrac{2}{x^2 + 4}$.

19. $x = \dfrac{1}{2}\csc y,\ \dfrac{-1}{x\sqrt{4x^2 - 1}}$.

Page 160, 161, Ex. 42

1. $6/\sqrt{1 - 9x^2}$.

3. $-2/(x^2 - 2x + 5)$.

5. $2x/\sqrt{1 - x^4}$.

7. $1/2\sqrt{-x^2 - x}$.

9. $\operatorname{arccot}(x/2) - \dfrac{2x(x^2 + 5)}{(x^2 + 4)^2}$.

11. $(-2,\ \pi - 2)$ max., $(2,\ 2 - \pi)$ min.

13. $2x^2/\sqrt{9 - x^2}$.

15. $\arcsin x$.

17. $2x\arctan x$.

19. $a/\sqrt{a^2 - x^2}$.

21. $-a/(a^2 + x^2)$.

27. $y/(x^2 + y^2 + x)$.

29. $(x^2 + y^2 + y)/x$,
    $2(y + 1)(x^2 + y^2)/x^2$.

31. $(-1,\ \pi - 2)$, $(3,\ 2 - \pi)$, $(1, 0)$,
    $x > 1$, $x < 1$.

33. $2/9$ rad./sec.

35. $2abw(5a - 3ab + 5c)/15$.

Page 164, Ex. 43

1. $50/3$ rads./min.

3. $12, 48$ rads.

5. 2 rads./unit time.

7. $4/5$ rads./sec.

9. $4x^2 + 9y^2 = 36$; 2, $(3, 0; \sqrt{26}/2)$, $(3\sqrt{3}/2, \sqrt{2})$.

11. 0.72 rads./hr.

13. $48\sqrt{5}/65$ rads.

15. $\sqrt{34}$, $8/17$.

Pages 165, 166, Ex. 44

1. 40π ft./sec.

3. $0, 0$; $2a\omega, 0$.

5. $\left(\dfrac{\pi a - 2a}{2},\ a\right)$.

9. $x = a\arccos\dfrac{a - y}{a} - \sqrt{2ay - y^2}$.

11. $x = 6(3t - \sin 3t)$, $y = 6(1 - \cos 3t)$.

13. $\omega\sqrt{a^2 + b^2 - 2ab\cos\theta}$.

Pages 168–170, Ex. 45

1. $-\dfrac{\cos 3x}{3} + C$.

3. $2\tan(x/2) + C$.

5. $\arctan 2x + C$.

7. $2\sin^2 x + C$.

9. $\arcsin 2x + C$.

11. 2.

13. $\dfrac{\sin^2 3x}{6} + C$.

15. $-3\tan(x/3) - x + C$.

17. $\dfrac{x}{2} - \dfrac{\sin 2x}{4} + C$.

19. $\dfrac{\sin^2 2x}{4} + C$.

21. 6.

23. $1/2$.

25. π.

27. $x = 2\cos t$, $y = 2\sin t$; $2, 1$; 2.

29. 4 rads.

31. $\sqrt{2}/4$, $-\sqrt{2}/4$.

33. $40/3$, $120/109$ rads.

35. $22/225$ rads /sec.

37. $\pi(18 - 8\sqrt{2} - \pi)/8$.

39. $\dfrac{2\sqrt{x + 1}\cos x}{2\sqrt{x + 1} + 1}$.

Pages 174, 175, Ex. 47

1. 2.

3. -2.

5. 3.

7. 72.

17. $\dfrac{\log \arctan y}{\log 2}$.

19. $-\dfrac{\log \arccos y}{\log 2}$.

21. arcsin 10^y.

23. $2 \log_2 y$.

25. $\pm 4/5$.

Pages 179, 180, Ex. 48

1. $2x \log_2 e/(x^2 + 1)$

3. $2x2^{x^2} \ln 2$.

5. $1/(x^2 - 4)$.

7. $e^{x/a} - e^{-x/a}$.

9. $2(e^{2x} - e^{-2x})/(e^{2x} + e^{-2x})$.

11. $-2 \sec x$.

13. $(1 - \ln x)/x^2$.

15. $4 \csc 2x \log e$.

17. $y = ex$.

19. $x - y - 1 = 0$.

21. $x - 4y = 4(1 - \ln 4)$.

23. $8x \ln 2 - y = 24 \ln 2 - 8$.

25. $x^x(1 + \ln x)$.

27. $(2/x) \ln 2x$.

29. $2e^{2x} e^{\sin e^{2x}} \cos e^{2x}$.

31. $1/(x \ln x)$.

33. $-2e^{3\theta+1}$.

35. $\dfrac{2}{x}(x + 1) \ln (xe^x)$.

37. $-1/(x \ln^2 ax)$.

39. $\dfrac{y(1 - xe^{x+y})}{x(1 + ye^{x+y})}$.

41. $\dfrac{2xe^{x^2-y^2} - ay}{2ye^{x^2-y^2} + ax}$.

Page 181, Ex. 49

1. $(-1, -1/e)$ min., $(-2, -2/e^2)$ infl., $x > -2$ upward.

3. $(0,0)$ min.

7. $a^4 e^{ax}$.

13. $2^x/\ln^5 2$.

15. 0.698.

17. $6\sqrt{3}\, x - 6y = 6 \ln 2 + \sqrt{3}\, \pi$.

Pages 183, 184, Ex. 50

1. $\dfrac{\ln x}{2} + C$.

3. $2^{x+1}/\ln 2 + C$.

5. $e^{x-1} + C$.

7. $\dfrac{\ln (x^2 - 1)}{2} + C$.

9. $\ln 16 + 2$.

11. $e^4 - 1$.

13. $2^{x^2}/2 \ln 2 + C$.

15. $(1/2) \ln (x^2 + 6x + 1) + C$.

17. $-\cos e^x + C$.

19. $(e^4 - 1)/e$.

21. $8(e^2 - 1)/e$.

23. $x^3 = 2y$.

25. $\pi(e^{12} - 1)/4$.

27. $\pi(8 \ln 2 + 6)$.

29. $-2 \log_b e/x\sqrt{x + 4}$.

31. e^{e^x+x}.

33. $-xb^{\sqrt{4-x^2}} \ln b/\sqrt{4 - x^2}$.

Pages 185–187, Ex. 51

1. $y = 20e^{10x}$.

3. $9y = 16(3/2)^t$.

5. 125,000.

7. 26.4 yrs.

9. 2.34 mins.

11. $y = 3e^{1-x/2}$.

19. $(\pi/4, 3\pi/4)$.

21. $\Delta x/x$.

25. (e,e), $(e^2, e^2/2)$.

Pages 191–193, Ex. 52

1. $(x^4 - 1)/2x^2 + \ln x^2 + C.$
3. $(x^3 + 2)^5/15 + C.$
5. $2\sqrt{x^3 + 2}/3 + C.$
7. $\sqrt{x^2 + 2x} + C.$
9. $(1/12) \ln (4x^3 - 3) + C.$
11. $-3(4 - x^2)^{2/3}$
13. $2 \arcsin x + C.$
15. $\dfrac{-1}{12(3 + 4x^3)} + C.$
17. $(-1/6) \cos^3 2x + C.$
19. $(1/2a) \sec^2 ax + C.$
21. $(1/15) \sin^5 3x + C.$
23. $(-1/2b) \csc^2 bx + C.$
25. $(1/a) \ln (1 + \tan ax) + C.$
27. $(-1/3) (1 - e^x)^3 + C.$
29. $1/(1 - e^x) + C.$

31. $x + 4 \ln (x - 1) + C.$
33. $x^2/2 + 3x + 5 \ln (x - 3) + C.$
35. $x^2/2 - (1/2) \ln (x^2 + 2) + C.$
37. $(3/2) \arctan 2x + C.$
39. $(1/5) (2x^{3/2} + 1)^{5/3} + C.$
41. $\dfrac{-1}{4(2x^2 + 1)} + C.$
43. $(6/\sqrt{7}) \arctan \sqrt{7}\, x + C.$
45. $(1/3a)(3 \tan ax + \tan^3 ax) + C.$
47. $\arcsin e^x + C.$
49. $(1/3)(\cos^3 x - 3 \cos x) + C.$
51. $\ln (e^x - e^{-x}) + C.$
53. $(1/a) \ln (e^{ax} - \sin ax) + C.$
55. $(1/6) \sin^3 x^2 + C.$
57. $(2/3) (1 + \ln x)^{3/2} + C.$
59. $\ln (e^x - e^{-x}) + C.$

Page 194, Ex. 53

1. $(1/3) \arcsin (3x/4) + C.$
3. $(1/\sqrt{2}) \arcsin (\sqrt{2}\, x/\sqrt{3}) + C.$
5. $\arctan (x + 1) + C.$
7. $\arcsin (x - 1) + C.$
9. $\arcsin (x/2) - \sqrt{4 - x^2} + C.$
11. $(1/4a)e^{4ax} + C.$
13. $-\ln (1 + \cos x) + C.$
15. $\ln\sqrt{x^2 + 4x + 2} + C.$
17. $\ln\sqrt{x^2 - 4x + 7} +$
    $\dfrac{10}{\sqrt{3}} \arctan \dfrac{x - 2}{\sqrt{3}} + C.$

19. $-\sqrt{3 + 2x - x^2} + C.$
21. $\dfrac{-\sqrt{2 - 3x^2}}{3} - \dfrac{2}{\sqrt{3}} \arcsin \dfrac{\sqrt{3}\, x}{\sqrt{2}} + C$
23. $x + 2\sqrt{3} \arctan (2x + 1)/\sqrt{3} + C.$
25. $(2\sqrt{3}/3) \arcsin (\sqrt{3}\, e^{x/2}/2) + C.$
27. $\ln \sin x + C.$
29. $(-1/\ln a)a^{\cos x} + C.$
31. $(3 \csc 4x - \csc^3 4x)/12 + C.$

Pages 197, 198, Ex. 54

1. $(1/3) \cos^3 x - \cos x + C.$
3. $x/2 - (1/8) \sin 4x + C.$
5. $(1/6) \cos^6 x - (1/4) \cos^4 x + C.$
7. $x + \cos x + C.$
9. $x/8 - (1/16) \sin 2x + C.$
11. $3x/8 - (1/4) \sin 2x + (1/32) \sin 4x + C.$
13. $\sqrt{2} \sin x + C.$
15. $(\sqrt{2}/3) \ln (\sec 3x/2 + \tan 3x/2) + C.$
17. $\ln (\csc x - \cot x) + C.$
19. $(1/35) \cos^5 x(5 \cos^2 x - 7) + C.$
21. $(3 \sin^2 x - 1)/3 \sin^3 x + C.$
23. $-\ln \sin x - (1/2) \csc^2 x + C.$

33. $-\cos e^x + C.$
35. $-e^{1/x} + C.$
37. $(1/8 \ln 2)2^{4x^2} + C.$
39. $(1/2) \arctan^2 x + C.$

25. $\cos x(5 \cos^6 x - 21 \cos^4 x + 35 \cos^2 x - 35)/35 + C.$
27. $\tan^5 x(5 \tan^2 x + 7)/35 + C.$
29. $(84x - 48 \sin 2x + 3 \sin 4x - 4 \sin^3 2x)/192 + C.$
31. $(24x - 8 \sin 4x + \sin 8x)/1024 + C.$

Pages 201, 202, Ex. 55

1. $2(x + 1)^{3/2}(3x - 2)/15 + C.$
3. $\arctan (\sqrt{x - 4}/2) + C.$
5. $3(x + 1)^{2/3}(2x - 3)/10 + C.$
7. $x/4\sqrt{4 - x^2} + C.$
9. $x - 2 \arctan (x/2) + C.$
11. $2\sqrt{x + 3} + C.$
13. $\ln (x + 1) + (4x + 3)/2(x + 1)^2 + C.$
15. $2x^{3/2}/3 - 3x + 18\sqrt{x} - 54 \ln (\sqrt{x} + 3) + C.$
17. $(1/a) \operatorname{arcsec} (x/a) + C.$
19. $-x/9\sqrt{x^2 - 9} + C.$
21. $4(1 + \sqrt{x})^{3/2} (3\sqrt{x} - 2)/15 + C.$
23. $-2 \cos \sqrt{x} + C.$
25. $2\sqrt{x^3 - 4}/3 - (4/3) \operatorname{arcsec} (x^{3/2}/2) + C.$
27. $(x^3 - 8)^{5/3}(5x^3 + 24)/40 + C.$
29. $4(\sqrt{x} - 1)^{3/2}(3\sqrt{x} + 2)/15 + C.$
31. $-4\sqrt{1 - \sqrt{x}}(\sqrt{x} + 2)/3 + C.$
33. $\dfrac{1}{6} \ln \dfrac{x}{x + 6} + C.$
35. $-\sqrt{x}/9(9x + 4) + (1/54) \arctan (3\sqrt{x}/2) + C.$

Pages 204, 205, Ex. 56

1. $e^{2x}/2 + C.$
3. $\sin x - x \cos x + C.$
5. $e^{3x}(9x^2 - 6x + 2)/27 + C.$
7. $x^3(3 \ln x - 1)/9 + C.$ ✓
9. $(x^2 \arctan x + \arctan x - x)/2 + C.$
11. $x^3(3 \ln x^2 - 2)/9 + C.$
13. $(1/2)(x^2 + 1) \arctan^2 x - x \arctan x +$
$\qquad \ln \sqrt{x^2 + 1} + C.$
15. $e^x(\sin x + \cos x)/2 + C.$
17. $-e^{-x^2}(x^2 + 1)/2 + C.$
19. $\ln \ln x + C.$

21. $2(\sqrt{x} + e^{\sqrt{x}}) + C.$
23. $(e^{2ax} - 4ax - e^{-2ax})/2a + C.$
25. $-(4 - x^2)^{3/2}/12x^3 + C.$
27. $-e^{-x}(\sin x + \cos x)/2 + C.$
29. $-e^{-x}(x^2 + 2x + 2) + C.$
31. $-(x \arctan \sqrt{x} + \sqrt{x} +$
$\qquad \arctan \sqrt{x})/2x + C.$
33. $-e^{-x}(\sin 3x + 3 \cos 3x)/10 + C.$
35. $-\dfrac{\cos x}{2 \sin^2 x} + \dfrac{1}{4} \ln \dfrac{1 - \cos x}{1 + \cos x} + C.$

Page 209, Ex. 57

1. $\ln (x - 2)^3/x^2 + C.$
3. $(1/2) \ln (3x + 1)^5/(x - 1) + C.$
5. $(1/2) \ln (x - 2)(x + 3)^4/(x + 2)^3 + C.$

7. $(1/2) \ln x/(x + 4) - 6/x + C.$

9. $(2x - 1)/2x^2 + \ln x/(x + 1) + C.$

11. $(1/2) \ln x^3/(x^2 + 1)^3 + \arctan x + C.$

13. $\ln (\sqrt{x} - 1)^2/\sqrt{x} + C.$

15. $x - \ln (x^2 + 2x + 10) - (8/3) \arctan (x + 1)/3 + C.$

17. $2 \arcsin (x - 1)/2 - \sqrt{3 + 2x - x^2} + C.$

19. $(1/8) \ln \tan^2 x/(\tan^2 x + 4) + C.$

21. $(1/2) \ln (1 - \sqrt{1 - x})/(1 + \sqrt{1 - x}) - \sqrt{1 - x}/x + C.$

23. $(2 - 3x)/(x - 1)(x - 2) + 2 \ln (x - 1)/(x - 2) + C.$

25. $(x - 1)(2 - x)/4(x^2 - 2x + 2) + (1/8) \ln x^2/(x^2 - 2x + 2) + (1/2) \arctan (x - 1) + C.$

27. $(1/8) \ln x^2/(x^2 + 1) + x^2/8(x^2 + 1) - (\ln x)/4(1 + x^2)^2 + C.$

Pages 211, 212, Ex. 58

1. $61/192.$

3. $\pi/3.$

5. $116/15.$

7. $1/3.$

9. $7/12.$

11. $\ln 3.$

13. $2.$

15. $(9\sqrt{3} - 10\sqrt{2})/24.$

17. $1.$

19. $(4 - \pi)\sqrt{2}/8.$

21. $\pi^2/4.$

23. $\pi/4.$

25. $16 - 12 \ln 3.$

27. $\pi ab.$

29. $2\pi.$

31. $(\pi + 6\sqrt{3} - 12)/12.$

33. $5/4.$

35. $\pi^2/2.$

37. $\ln (4/3).$

39. $5\pi^2 a^3.$

41. $\dfrac{2xe^{\arctan x^2}}{1 + x^4}.$

43. $3 \ln a \, a^{\ln \sin 3x} \cot 3x.$

49. $30\sqrt{3}$ ft.

51. $\pi(5e^4 - 4e^2 - 1)/2e^4.$

53. $y = e^{(2x-1)/x}$

55. 3 ft.-lbs.

Pages 216, 217, Ex. 59

1. $1.$

3. None.

5. None.

7. None.

9. $\pi/8.$

11. None.

13. None.

15. None.

17. None.

19. None.

21. None.

23. $\dfrac{3}{10}\left(\dfrac{1}{\sqrt[3]{289}} - \dfrac{1}{\sqrt[3]{9}}\right).$

25. None.

27. 2, none.

29. None, $1/8.$

31. None.

33. $2\pi.$

35. None.

37. None.

39. $\pi/2.$

Page 219, Ex. 60

19. $e^{\sin^2 x} + C.$

21. $x/8(x^2 + 4) + (1/16) \arctan (x/2) + C.$

23. $2(a^{1/3} + x^{1/3})^{2/3}(15x^{2/3} - 12a^{1/3}x^{1/3} + 8a^{2/3})/35 + C.$

25. $-\arctan (\cos x) + C.$

27. $(1/12) \ln (3 + 2 \sin x)/(3 - 2 \sin x) + C.$

29. $\sin 3x(3 \cos^4 3x + 4 \cos^2 3x + 8)/45 + C.$

31. $-(105 \cos^3 3x - 189 \cos^5 3x + 135 \cos^7 3x - 35 \cos^9 3x)/945 + C.$

33. $(-5 \csc 2x - 15x + 5 \sin^3 2x - \sin^5 2x)/10 + C.$

35. $(3 \tan 8x + \tan^3 8x)/24 + C.$

37. $(1/2a) \ln (a - e^x)/(a + e^x) + C.$

Answers

Pages 225–227, Ex. 61

1. $16/125$.

3. $\sqrt{2}/2$.

5. $2\sqrt{5}/25a$.

7. $\sqrt{2}$.

9. $24/125$.

11. $\left(-x^3/4, \dfrac{8 + 3x^2}{4}\right)$.

13. $1/e$.

15. $\sqrt{a}/2(x + y)^{3/2}$.

17. $3\sqrt{21}/98$.

19. ∞.

21. $4\sqrt{2}/\pi$.

23. none.

25. $(-7/6, 3)$.

27. $(-\ln \sqrt{2}, \sqrt{2}/2)$.

29. $(0, 0)$.

31. $a\theta$.

33. $16\sqrt{2}/\pi^3$.

35. $2(1 + \cos^2 \theta)^{3/2}$.

37. $\sqrt{2}$.

39. $27ak^2 = 8(h - a)^3$.

Pages 228, 229, Ex. 62

1. 4π.

3. $\ln (\sqrt{2} + 1)$.

5. $3a/2$.

7. $6a$.

9. $\sqrt{5} - \sqrt{2} + \ln (2\sqrt{2} + 2)/(\sqrt{5} + 1)$.

11. $\ln (3/4) - 1/2$.

13. $(3 + 8 \ln 2)/8$.

15. $e^{a/2} - e^{-a/2}$.

17. $\sqrt{2}(1 - e^{-\pi})$.

19. 4π.

Pages 231, 232, Ex. 63

1. $4\pi a^2$.

3. $2\pi[\sqrt{2} + \ln (1 + \sqrt{2})]$.

5. $12\pi a^2/5$.

7. $4\pi^2 a^2$.

9. $2\pi(2a + e^a - e^{-a})$.

11. 60π.

13. $64\pi a^2/3$.

15. $\pi a^2 [2\sqrt{7} - 1 - \dfrac{\sqrt{2}}{2} \ln (2\sqrt{2} + \sqrt{7})/(\sqrt{2} + 1)]$.

Pages 233. 234, Ex. 64

1. $\pi a/4$.

3. $8/3$.

5. 20.

7. $8\sqrt{3}/5$.

11. $2a^2/\pi$.

13. $\pi a^3/2$.

15. $2\sqrt{2bg}/3$.

17. $(64 \ln 2 - 15)/12$.

19. $k(t_2^2 + t_1^2)(t_2 + t_1)$.

Page 238, Ex. 65

1. $16/5$.

3. $3\pi a^2$.

5. $\sqrt{2} - 1$.

7. $32/3$.

9. $2a^2(3\pi + 8)/3$.

11. $(2e - 3)/e^2$.

13. $2a^2 - a\sqrt{5}$.

15. $12 \arcsin (\sqrt{2}/\sqrt{3}) - 2\sqrt{2}/5$.

17. $2\pi ab$.

19. $\pi[\sqrt{2} + \ln (1 + \sqrt{2})]$.

Pages 241, 242, Ex. 66

1. $4\pi a^3/3$.
3. $512\pi/5$.
5. $52\pi/27$.
7. $512\pi\sqrt{2}$.
9. $\pi/4$.
11. $11\pi/6$.
13. $2\pi^2 a^3$.

15. $4\pi(2\ln 2 - 1)$.
17. $4\pi a^2 b/3$.
19. $\pi(8\ln 2 - 3)/2$.
21. $a^3\pi w$.
23. $11,875\pi w/12$ ft.-lbs.
25. $2a^2/3$.

Page 245, Ex. 67

1. $(9/4, 3)$.
3. $(1.82, 0.598)$.

5. No real solution.
7. ± 1.

Page 248, Ex. 68

1. 1.
3. 0.
5. -1.

7. 0.
9. e.
11. $1/3$.

13. 0.
15. 2.
17. e^2.

Pages 251, 252, Ex. 69

1. $1/n^2$.
3. $1/2^{n-1}$.
5. $n/2^{n-1}$.
7. $1/n(n+1)$.
9. $1 + 2/3 + 1/3 + 2/15 + \cdots$.
11. $1 - 4/5 + 3/5 - 8/17 + \cdots$.

13. 145.
15. 2.
17. $1/(n^2 + 1)$.
19. $n(n+1)/(n+2)^3$.
21. $n(a+l)/2,\; a+(n-1)d$.

Pages 255, 256, Ex. 70

1. Convergent.
3. Convergent.

5. Divergent.
7. Convergent.

9. Convergent.
11. Convergent.

Page 260, Ex. 71

1. Convergent.
3. Convergent.

5. Divergent.
7. Convergent.

9. Convergent.

Page 262, Ex. 72

1. $-1 \leqq x < 1$.
3. $-1 \leqq x < 1$.

5. All values.
7. $-1 \leqq x < 1$.

9. $x \leqq -1,\; x > 1$.
11. $1 < x < 3$.

Page 265, Ex. 73

7. $\ln a + \dfrac{x}{a} - \dfrac{x^2}{2a^2} + \dfrac{x^3}{3a^3} + \cdots$.

9. $1 + \dfrac{x^2}{2} - \dfrac{x^4}{2\cdot 4} + \dfrac{3x^6}{2\cdot 4\cdot 6} + \cdots$.

11. $1 + \dfrac{x^2}{2!} + \dfrac{x^4}{4!} + \dfrac{x^6}{6!} + \cdots$.

13. $x - \dfrac{x^2}{2} + \dfrac{x^3}{6} - \dfrac{x^4}{12} + \cdots$.

15. $1 - 2\dfrac{x^2}{2!} - 2^3\dfrac{x^4}{4!} - 2^5\dfrac{x^6}{6!} + \cdots$.

17. $\dfrac{1}{2}\left(1 - \sqrt{3}\,x - \dfrac{x^2}{2!} + \dfrac{\sqrt{3}\,x^3}{3!} + \cdots \right)$.

19. $1 + \dfrac{x^2}{2!} + \dfrac{5x^4}{4!} + \dfrac{61x^6}{6!} + \cdots$.

21. $x - \dfrac{x^3}{2\cdot3} + \dfrac{3x^5}{2\cdot4\cdot5} - \dfrac{3\cdot5x^7}{2\cdot4\cdot6\cdot7} + \cdots$.

23. $x^2 - \dfrac{x^4}{3} + \dfrac{x^6}{5} - \dfrac{x^8}{7} + \cdots$.

25. $1 - \dfrac{x^4}{2!} + \dfrac{x^8}{4!} - \dfrac{x^{12}}{6!} + \cdots$.

27. $1 + \dfrac{x^4}{2} + \dfrac{3x^8}{2\cdot4} + \dfrac{3\cdot5x^{12}}{2\cdot4\cdot6} + \cdots$.

Pages 268, 269, Ex. 74

1. 2.7183.

3. 0.0523.

5. 0.1823.

7. $2\left(x + \dfrac{x^3}{3} + \dfrac{x^5}{5} + \dfrac{x^7}{7} + \cdots \right)$.

9. 3.142.

11. 0.52992.

15. 0.36434.

17. 0.22 or 12° 36′.

19. 0.1987

21. 1.

23. 0.

25. 0.7467.

Pages 270, 271, Ex. 75

1. $e^4\left[1 + (x-4) + \dfrac{(x-4)^2}{2!} + \dfrac{(x-4)^3}{3!} + \cdots \right]$.

3. $\dfrac{1}{2}\left[1 - \sqrt{3}\left(x - \dfrac{\pi}{3} \right) - \dfrac{1}{2!}\left(x - \dfrac{\pi}{3} \right)^2 + \dfrac{\sqrt{3}}{3!}\left(x - \dfrac{\pi}{3} \right)^3 + \cdots \right]$.

5. $1 + 2\left(x - \dfrac{\pi}{4} \right) + \dfrac{4}{2!}\left(x - \dfrac{\pi}{4} \right)^2 + \dfrac{16}{3!}\left(x - \dfrac{\pi}{4} \right)^3 + \cdots$.

7. $\sqrt{2}\left[1 + \dfrac{x-1}{2} + \dfrac{(x-1)^2}{8} - \dfrac{(x-1)^3}{16} + \cdots \right]$.

9. $\ln 2 + \dfrac{(x-2)}{2} - \dfrac{(x-2)^2}{2^2\cdot2!} + \dfrac{2(x-2)^3}{2^3\cdot3!} + \cdots$.

11. $(x-1)^3 + (x-1)^2 - 2(x-1) - 7$.

13. $-\dfrac{1}{2}\ln 2 + \left(x - \dfrac{\pi}{4} \right) - \left(x - \dfrac{\pi}{4} \right)^2 + \dfrac{2}{3}\left(x - \dfrac{\pi}{4} \right)^3 + \cdots$.

17. $e^a\dfrac{(x-a)^{n-1}}{(n-1)!}$.

Page 272, Ex. 76

1. 0.8746.

3. 0.5736.

5. 1.9680.

7. 0.85723.

9. 3 terms.

11. 2.01545.

Pages 274, 275, Ex. 77

13. $\rho^2 \sin 2\theta = 4$.

15. $\rho = a$.

17. $y - a = 0$.

31. $\rho = 2a \,(\cos \theta - \sin \theta)$.

33. $\rho(\cos^3 \theta + \sin^3 \theta) = 3a \sin \theta \cos \theta$.

35. $(x^2 + y^2)^2 = a(x^2 - y^2)$.

43. $(2, 0)$.

45. $(0, 0)$, $(a/\sqrt[4]{2}, \pi/4)$.

     $(a/\sqrt[4]{2}, 5\pi/4)$.

47. $\rho^2 = a^2 \,(\sin^6 \theta + \cos^6 \theta)$.

Page 277, Ex. 78

1. $\pi/2$.

3. $\arctan (1/a)$.

5. 0.

7. $\arctan (1 - \sqrt{2}) = 157° \, 30'$.

9. $\pi/4$.

11. 3.

13. $\pi/2$.

15. $\arctan (\sqrt{3}/2)$.

17. $-5/12$.

Pages 278, 280, Ex. 79

1. πa^2.

3. $19\pi/2$.

5. a^2.

7. $\pi a^2/4$.

9. $4\pi a^3/3$.

11. $\pi a^2/4$.

13. a^2.

15. $(6 - \pi)/3$.

17. $(2\pi - 3\sqrt{3})/2$.

19. $8a^2/3$.

21. $8\pi/3$.

23. $\rho = A e^{\theta/k}$.

Page 281, Ex. 80

1. $2\pi a$.

3. $3\pi a/2$.

5. $4\pi a^2$.

7. $3\pi/2$.

9. $\sqrt{1 + a^2}(e^{a\theta_2} - e^{a\theta_1})/a$.

11. $a^2(8\pi - 9\sqrt{3})/16$,

     $a^2(16\pi + 9\sqrt{3})/16$.

Page 283, Ex. 81

1. $9\pi a^2$.

3. $3\pi/2$.

5. $a^2(3 \arctan 2 + 2 - \pi)/4$.

7. $a^2(\pi - 1)$.   **9.** $4\pi a^3/3$.

11. $a^2(2\pi + 3\sqrt{3})/3$.

13. $a^2(9\pi + 16)/12$.

Page 287, Ex. 82

1. $x = 0$, $y = 0$, $z = 0$.

3. $x \pm y = 0$.

5. $y \pm z = 0$.

7. 6.

9. $\sqrt{86}$.

11. $bx + ay = ab$.

13. $cy + bz = bc$.

15. $\rho^2 + z^2 = a^2$, $r = a$.

17. $z = x^2 + y^2$, $x^2 + y^2 = 2ay$,

     $z = 2a - x$.

19. $z = a e^{\rho^2}$.

1. $\sqrt{17}, \sqrt{6}, 5.$

3. $1, -3, -1; /\sqrt{11}.$

5. No

7. $\left(\dfrac{5}{2}, \dfrac{5}{2}, \dfrac{\pm 5\sqrt{2}}{2}\right), \pm \dfrac{\sqrt{2}}{2}.$

9. $\dfrac{21\sqrt{10}}{130}, \dfrac{-11\sqrt{13}}{130}, \dfrac{33\sqrt{130}}{650}.$

11. $54° \; 44.1'.$

15. $1, -5, -2.$

1. $3, \frac{1}{3}, \frac{2}{3}, -\frac{2}{3}; \; 4, -\frac{2}{3}, \frac{1}{3}, -\frac{2}{3};$
$\quad 2, -\frac{3}{5}, \frac{4}{5}, 0.$

3. $2x + y + 2z = 3$

7. $3x + 6y + 2z = 6.$

9. $2x + 2y - z = 3.$

13. $3x - 4y - 6z = 12.$

15. $2x - 4y - 3z + 9 = 0$

19. $3, 5, 6$

1. $1/2, 1/2, \sqrt{2}/2; \; 60°, 60°, 45°.$

3. $x/2 = (y - 1/3)/2 = (z - 5/6)/1,$
$\quad 2/3, 2/3, 1/3$

7. $x - 4 = (y + 3)/-5 = z - 2.$

9. $(x + 2)/\sqrt{11} = (y - 3)/2$
$\quad = (z + 4)/1.$

11. $3y + 4z = 6, x = 0; \; x + 2z = 3,$
$\quad y = 0; \; 2x + 3y = 8, z = 0.$

1. $(x - h)^2 + (y - k)^2 + (z - l)^2 = a^2$

3. Cylinders: elliptic, parabolic, hyperbolic, circular, parabolic.

5. $b^2 x^2 + a^2(y^2 + z^2) = a^2 b^2.$

7. $z^4 = 16(x^2 + y^2)$

9. $c^2(x^2 + y^2) - a^2 z^2 = a^2 c^2.$

11. $(a^2 - c^2)x^2 + a^2(y^2 + z^2)$
$\quad = a^2(a^2 - c^2).$

13. $x^2 + y^2 = 4pz.$

15. $(3z - 4)^2 = 4(x^2 + y^2 - 10y + 25).$

17. $x^2 + y^2 = 16, (0, 0, 2), (0, 0, 4).$

19. Locus of focus $x^2 + z^2 = 5,$
$\quad (\pm 2\sqrt{10}/3, 0, 0).$

1. $x^2 + y^2 = 9z^2, x^2 + y^2 = 16(z - 1)^2.$

3. $(0, 0), x = \pm 2z, y = \pm 2z.$

5. $9x^2 + 4y^2 = 36, 4x^2 + z^2 = 16,$
$\quad 16y^2 + 9z^2 = 144.$

7. $x^2 - y^2 = 4, 9x^2 + 4z^2 = 36,$
$\quad 9y^2 - 4z^2 + 36 = 0.$

9. $x^2 - y^2 = 4, 9x^2 - 4z^2 = 36.$

11. $256\pi/3.$

13. $4\pi abc/3.$

15. $25\pi/2.$

1. Ellipsoid.

3. Hyperboloid-two nappes.

5. Elliptic paraboloid.

7. Hyperbolic cylinder.

9. Sphere.

11. Prolate spheroid.

13. Elliptic cone.

15. Hyperboloid-one nappe

17. Elliptic paraboloid.

19. Paraboloid.

21. Hyperboloid-one nappe.

23. Circular cone.

25. Circular cylinder.

27. Circular cylinder.

29. Parabolic cylinder.

31. Circular cylinder.

33. $x^2 + y^2 + z^2 = 25.$

35. $9y^2 - 4x^2 + 4z^2 = 36.$

37. $500\pi/3.$

39. $2\pi.$

Page 314, Ex. 89

1. $4x$, $6y$.

3. $\dfrac{-x}{\sqrt{4-x^2-y^2}}$, $\dfrac{-y}{\sqrt{4-x^2-y^2}}$.

5. $-y/(x-y)^2$, $x/(x-y)^2$.

7. ye^{xy}, xe^{xy}.

9. 2, 1.

11. $ye^{xy}(xy+1)$, $xe^{xy}(xy+1)$.

13. $e^\theta \sin 2\phi$, $2e^\theta \cos 2\phi$.

15. $e^x[\cos(x-y) + \sin(x-y)]$, $-e^x \cos(x-y)$.

17. $2xe^{x^2+y^2}$, $2ye^{x^2+y^2}$.

19. $(3, 9/4, 15/4)$, $4/5$.

21. $2x + y + 2x = 9$.

Pages 317, 318, Ex. 90

1. $3x - 2y + 2z = 17$.

3. $4x + 3y - 3z + 4 = 0$.

5. $4x + 2y + z = 12$.

11. $x - y + z + 1 = 0$.

13. $x_1x + y_1y + z_1z = a^2$.

15. $b^2c^2x_1x - a^2c^2y_1y \pm a^2b^2z_1z = a^2b^2c^2$.

17. $-2xy/(x^2+y^2) + 2\arctan(y/x)$, $2xy/(x^2+y^2) + 2\arctan(y/x)$, $(x^2-y^2)/(x^2+y^2)$.

19. $4x + 4y - z = 6$.

21. $-v/p$, $-p/v$; k/v, v/k; k/p, p/k.

Pages 321, 322, Ex. 91

1. $(2x+y)\,\Delta x + (x+2y)\,\Delta y + (\overline{\Delta x}^2 + \Delta x\,\Delta y + \overline{\Delta y}^2)$.

3. $\overline{\Delta x}^2 + \Delta x\,\Delta y + \overline{\Delta y}^2$.

5. 7.1407.

7. 19.0122.

9. $\dfrac{2(x\,dy - y\,dx)}{(x-y)^2}$.

11. $e^{xy}(y\,dx + x\,dy)$.

13. 128.320.

15. 2.505.

17. 0, 0.

19. 0.03.

21. 0.199 sq. in.

Page 324, Ex. 92

3. $3x + 2y - 6z = 49$.

7. $\dfrac{-z}{x + y\sin yz}$, $\dfrac{-z\sin yz}{x + y\sin yz}$.

5. $4x + y = 3z$.

9. $-\dfrac{ye^{xy} + ze^{xz}}{xe^{xz} + ye^{yz}}$, $-\dfrac{xe^{xy} + ze^{yz}}{ye^{yz} + xe^{xz}}$.

Page 330, Ex. 93

1. -6.

3. $11/84$.

5. 12.

7. 16π.

9. 8π.

11. $81\pi/2$.

13. 4π.

15. $1664/35$.

17. $16a^3/3$.

19. $569/140$.

Page 333, Ex. 94

1. $\pi a^2 h$.

3. $64a^3/9$.

5. $2\pi a^3/3$.

7. $a^3/18$.

9. $4\pi a^3(8 - 3\sqrt{3})/3$.

11. $4\pi a^3/7$.

Answers

Page 338, Ex. 95

1. $81/4$.
3. $(56 - \ln 8)/18$.
5. $\pi/2$.
7. $1/4$.

9. $\pi a^3/2$.
11. $4\pi abc/3$.
13. $100\pi/3$.
15. $2\pi(23\sqrt{23} - 101)/3$.

Page 343, Ex. 96

1. $\pi k a^4/4$.
3. $\pi k a b^3/4$, $\pi k a^3 b/4$.
5. $4k/9$.
7. $64k/63$.

9. $22k\pi$.
11. $3\pi k a^4$.
13. $a^4 k(320 + 81\pi)/96$.

Page 346, Ex. 97

1. $(2b/3, a/3)$.
3. $(4a/3\pi, 4b/3\pi)$.
5. $(a/5, a/5)$.
7. $(16/7, 0)$.
9. $(\pi a, 5\pi/6)$.

11. $\left(\dfrac{2a}{12 - 3\pi}, \dfrac{2a}{12 - 3\pi} \right)$.
13. $(1, \pi/8)$.
15. $(256\sqrt{2}\, a/105\pi, 0)$.

Page 347, Ex. 98

1. $(0, 2a/\pi)$.
5. $(2a/5, 2a/5)$.

3. $\left(0, \dfrac{a(e^4 + 4e^2 - 1)}{4e(e^2 - 1)} \right)$.

Page 348, Ex. 99

1. $(a/4, b/4, c/4)$.
3. $(3a/8, 3b/8, 3c/8)$.

5. $(0, 0, 4/3)$.
7. $\bar{z} = 3\sqrt{2}/16$.

Page 353, Ex. 100

1. $k\lambda\sqrt{2}/a$.
3. $2k\lambda\pi$.

5. $2k\lambda\pi a(\sqrt{2} - 1)/3$.
7. $k\lambda\pi[3\sqrt{3} - 3\ln(2 + \sqrt{3})]$.

Page 359, Ex. 101

1. $x^2 + y^2 = a^2$.
5. $(3y^2 - 2y)\dfrac{dy}{dx} + 2x = 0$.
9. $xy + x + 2y = C$.

11. $\ln y + x - \sin x = C$.
13. $y = C - x \pm x^2$.
15. $x\dfrac{dy}{dx} - y = 0$.

Pages 360, 361, Ex. 102

1. $x^2 + 4\ln y = C$, $y^4 = Ae^{-x^2}$.
3. $\ln(1 + y^2) + 2\arctan x = C$.
5. $y(1 + x^2)(1 + y) = A(1 - y)$.
7. $y\dfrac{dy}{dx} + x = 0$.
9. $x\dfrac{dy}{dx} - 2y = 0$.

11. $2xy\dfrac{dy}{dx} + x^2 - y^2 = 0$.
13. $y\left(\dfrac{dy}{dx}\right)^2 - 2x\dfrac{dy}{dx} + y = 0$.
15. $y = x\dfrac{dy}{dx} \pm 3\sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}$.

Pages 363, 364, Ex. 103

1. $xy^2 = C(2y + x)$.

3. $\dfrac{2y + (1 - \sqrt{3})x}{2y + (1 + \sqrt{3})x}(2y^2 + 2xy - x^2)^{\sqrt{3}} = C$.

5. $\ln x + e^{-y/x} = C$.

7. $y = Ax\sqrt{x^2 + y^2}$.

9. $(x^2 - 4)\,dy - xy\,dx = 0$.

11. $y\sqrt{x^2 + 1} = C$.

13. $xy = Ae^{x-y}$.

15. $\sec x + \tan y = C$.

Page 365, Ex. 104

1. $x^2 + 2xy - y^2 = C$.

3. $y(e^x - 3) = C$.

5. $3xy^2 - 4x^3 = C$.

7. $x^2 + y^2 + Cx = 0$.

9. $y(1 - x) = ax$.

11. $x^2 = C\sqrt{x^2 + y^2}$.

13. $\ln x + \arctan(y/x) = C$.

15. $y = e^x$.

Pages 367, 368, Ex. 105

1. $y = x^2 + ax$.

3. $ye^x = x + C$.

5. $e^{\tan x}(y - 1) = C$.

7. $y = x^2 + Cx^2e^{1/x}$.

9. $3y(x^2 + 1) = 4x^3 + A$.

11. $e^{\cos x}(\cos y - 1) = C$.

13. $x^3\sqrt{3 - y^2} = Ay$.

15. $y^3 = C(y^2 - x^2)$.

17. $(2y^2 - 1)e^{2x^2} = C$.

19. $x^3y^5(5 - Cx^2) = 2$.

21. $2xy^1 = \ln(x^2 - 1) + x^2 + C$.

23. $\tan x \tan y = A$.

Pages 370, 371, Ex. 106

1. $2y = 2x \ln x + x^2 + C_1x + C_2$.

3. $y = C_1(x^2 + 2x) + C_2$.

5. $4y = x^2 + C_1 \ln x + C_2$.

7. $4y = x^4 + C_1x^3 + C_2$.

9. $2y = \pm x\sqrt{x^2 + C_1}$
$\pm C_1 \ln(x + \sqrt{x^2 + C_1}) + C_2$.

11. $y(C_1x + C_2) = 1$.

13. $x^2y + 2x = Cy$.

15. $3x^2y = Cx^3 - 1$.

17. $x^2 + y^2 = a^2$.

19. $xy = a$.

Pages 374, 375, Ex. 107

1. $y = e^{2x}(A + Bx)$.

3. $y = e^{2x}(A \cos 3x + B \sin 3x)$.

5. $y = e^{-x/2}\left(A \cos \dfrac{\sqrt{3}x}{2} + B \sin \dfrac{\sqrt{3}x}{2}\right)$.

7. $y = Ae^{2x} + Be^{-2x}$.

11. $x = gt^2/2 + C_1t + C_2$.

13. $x = (1/k)\ln(kv_0t + 1)$.

15. 12,960 ft., 27.2 secs.

17. $y = Ae^x + Be^{2x} + Ce^{-x}$.

19. $y = e^x(A + Bx) + Ce^{2x}$.